AN AUGMENTATION APPROACH AND THEORY OF THE RESULTANT FOR LINEAR TIME-VARYING SYSTEMS

by

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In the first half of the dissertation, an algebraic approach is developed based on "augmentation" for linear discrete-time time-varying systems. Using this approach, it is shown how to design feedback control systems for a broad class of time-varying systems which result in a closed-loop system matrix that is equivalent to a constant matrix with arbitrary assignable eigenvalues. The theory yields controllers having a much lower order in general than that obtained in past work. In the second half of the dissertation, a discrete-time time-varying version of Sylvester's resultant matrix is defined. This definition is then used to check coprimeness of polynomials and polynomial matrices with time-varying coefficients, to examine reachability of a given system, and to design compensators which result in a type of assignability.

CHAPTER ONE
INTRODUCTION

Time-varying systems can be found in many practical applications, such as linearized models of nonlinear systems about a time-varying nominal trajectory, adaptive aircraft systems with time-varying parameters, integrated circuits with time-varying components, and so on. Thus, there is a good deal of motivation for developing the theory of linear time-varying systems.

The first part of this dissertation is concerned with developing the algebraic theory of linear time-varying systems. Our emphasis is on feedback control system design that is applicable to a broad class of time-varying systems.

A part of the past work on feedback control of linear time-varying systems is based on the minimization of a quadratic cost and is specified in terms of Riccati difference equations. For example, CHENG [1979] and KWON and PEARSON [1978] have obtained stabilizing memoryless feedback laws for uniformly reachable time-varying systems. HAGER and HOROWITZ [1976] and ANDERSON and MOORE [1981] have defined the notion of stabilizability for discrete-time, time-varying systems and have derived the optimal control in memoryless feedback form.

Other approaches to linear time-varying systems are based on algebraic formulations which yield systematic methods for computing

feedback control laws. For example, WOLOVICH [1968] has obtained a stabilizing feedback law for continuous time-varying systems possessing a fixed lexicographic basis. MORSE and SILVERMAN [1973] have derived coefficient-assignability results on the control of index-invariant continuous time-varying systems. These previous results can be easily extended to the discrete-time case. Using the concept of n-cyclicity, KAMEN and HAFEZ [1979] have developed an algebraic theory for discrete-time time-varying systems.

In general, approaches based on algebraic formulations provide more flexibility and insight than the one based on the minimization of a quadratic cost. But all the past work based on the algebraic approaches mentioned above treats very restricted classes of linear time-varying systems, such as index-invariant or n-cyclic systems. Therefore, it is desirable to develop an algebraic theory of feedback control for a much broader class of linear time-varying systems.

In time-varying systems, it is possible that the system is reachable at all times in N steps where N is strictly greater than the system dimension n. This is due to the lack of a "Cayley-Hamilton" type theorem. Therefore, it is not difficult to see that N-step reachability is a much weaker condition than n-step reachability which in turn is weaker than index-invariance and the existence of a fixed lexicographic basis. Moreover, N-step reachability is weaker than the notion of uniform reachability (see KWAKERNAAK and SIVAN [1972]).

KAMEN et al. [1985] and POOLLA [1984] have developed an algebraic theory for N-step reachable time-varying systems using a transfer-

function approach. In this previous work, the authors have successfully generalized polynomial matrix-fraction methods of the linear time-invariant system theory to the linear time-varying case, and have obtained coefficient-assignability results for N-step reachable discrete-time time-varying systems by dynamic state feedback. Time-varying polynomial matrix-fraction methods used in this work are based on a framework specified in terms of noncommutative (skew) rings of polynomials, formal series, and formal Laurent series, all with coefficients in a ring of time functions.

In this dissertation, we present a constructive method for finding a dynamic state feedback controller when the linear time-varying system is N-step reachable at all times. Although KAMEN et al. [1985] have obtained a dynamic state-feedback controller, we believe that our scheme for finding a controller is much more flexible. Moreover, our procedure yields a controller having a much lower order in general than the controller derived by KAMEN et al. [1985].

The second part of this dissertation is concerned with developing the theory of the resultant of linear time-varying systems.

The resultant is a classical method for determining the existence of a common factor of two time-invariant (i.e. constant-coefficient) polynomials. From the resultant, an algorithm for determining a greatest common divisor (gcd) of two polynomials has been derived (see BARNETT [1983]). There are several definitions for time-invariant polynomials; for example, Sylvester's resultant, Macduffee's resultant, and Bezout's resultant. For the definitions of these resultants, the reader may refer to KAILATH [1980] and BARNETT [1983].

The resultant was first generalized to the case of time-invariant polynomial matrices by ROWE [1972]. Later, ANDERSON and JURY [1976] and BITMEAD et al. [1978] have modified ROWE's resultant and obtained an algorithm to find a gcd of polynomial matrices.

The resultant has been used in system theory to obtain minimal state-space realization of transfer-function matrices, to study decoupling zeros and uncontrollable and unobservable modes of given systems, to obtain the pole-zero structure of given multivariable systems, and to design dynamic controllers (see CHEN [1984], BARNETT [1983], and KAILATH [1980]).

In the time-varying case, there is a notion of coprimeness of two time-varying polynomials and polynomial matrices. This notion is defined in terms of a time-varying version of the Bezout identity (see Ilchmann et al. [1984]). But, this definition of coprimeness is not related to a notion of the resultant. In this dissertation, we will define the discrete-time time-varying version of the resultant and we will find the connection between coprimeness and the resultant. This resultant can also be used to check reachability of a given time-varying system and to design feedback control systems.

In Chapter Two, we summarize basic definitions and results from KAMEN and HAFEZ [1979], KAMEN et al. [1985], POOLLA [1984], and so on. Those definitions and results include the skew polynomial ring structure, n-cyclic systems, N-step reachability, control-canonical form, and so on.

In Chapter Three, we present an algebraic approach based on "augmentation" for constructing a generalized control-canonical form for N-step reachable time-varying systems. If a given discrete-time time-varying system is N-step reachable at all times, then it is possible to turn the given system into a system having a fixed basis by simply augmenting the system state. This augmentation is always possible as a result of the special properties of the ring of time functions. From the fixed basis, a control-canonical form for the augmented system is constructed in this chapter, which we call the generalized control canonical form. Since the augmentation forces the system to have a fixed basis, the augmented system is index invariant, and the discrete-time versions of the results of WOLOVICH [1968] and MORSE and SILVERMAN [1973] can then be applied to N-step reachable systems. For example, the coefficients of the closed-loop characteristic polynomials of N-step reachable systems can be assigned arbitrarily.

The generalized control-canonical form leads to the design of a state feedback gain matrix for the augmented system which assigns the coefficients of the closed-loop system matrix arbitrarily. Therefore, the feedback gain matrix plus the augmentation comprises the controller for the N-step reachable system, and we have the result that we can construct a closed-loop system whose system matrix is algebraically equivalent to a constant matrix with arbitrary assignable eigenvalues. This assignability result, which was first obtained by KAMEN et al. [1985] and POOLLA [1984], is discussed in Chapter Four. Also, in this chapter, the augmentation approach is applied to design of output

feedback controllers when the system is given by an input-output difference equation. We note that a different augmentation approach in the time-invariant case was considered by BRASCH and PEARSON [1970]. The augmentation presented in this dissertation is an augmentation of system state, whereas the one considered by BRASCH and PEARSON is an augmentation of the system input and output.

In Chapter Five, cyclizability of N-step reachable time-varying systems is discussed. HEYMANN [1968] has developed a direct method to find a state-feedback matrix without using a control canonical form for multivariable time-invariant systems. Roughly speaking, the system is cyclizable if there is a feedback gain matrix so that the closed-loop system is reachable through one of the inputs. It is well known that a time-invariant system is cyclizable if and only if it is reachable. In the time-varying case, however, the notion of cyclizability has not been completely investigated. In this chapter, we shall discuss cyclizability of the augmented system for N-step reachable time-varying systems. This result is then applied to the construction of feedback control systems for N-step reachable time-varying systems.

Chapter Six treats the first half of the resultant theory for time-varying systems. In this chapter, we define a time-varying resultant matrix and this definition is used to check reachability of a given time-varying system. This chapter is divided into two parts. In the first part, we discuss the resultant matrix for the SISO case, and in this case, the resultant matrix resembles Sylvester's resultant. For the time-invariant case, the reader may refer to BARNETT [1983]. The

second part of this chapter generalizes the definition of the SISO resultant matrix to the MIMO case. The reader may refer to ROWE [1972], ANDERSON and JURY [1976], and BITMEAD et al. [1978] for the time-invariant MIMO case.

The second half of the resultant theory for time-varying systems, which includes the relationship between coprimeness and the resultant, design of compensators, and so on, is covered in Chapter Seven.

Finally, in Chapter Eight, we make some concluding remarks and discuss some open problems related to the topics covered in this dissertation.

CHAPTER TWO
BACKGROUND

The purpose of this chapter is to give various preliminary definitions and results. It is intended to help the reader in understanding the material covered in this dissertation.

With $\mathbf{Z}$ = set of integers and $\mathbf{R}$ = field of real numbers, let $\mathbf{A}$ denote the set of all functions defined on $\mathbf{Z}$ with values in $\mathbf{R}$. With pointwise addition and multiplication, A is a commutative ring with identity $\mathbf{1}$, where $\mathbf{1}(k) = 1$ for all k in $\mathbf{Z}$.

Let σ be the right-shift operator defined as

$$\begin{aligned} \sigma \ : \ A &\longrightarrow A \\ a(k) &\longmapsto (\sigma a)(k)=a(k-1), \text{ for all integers } k. \end{aligned}$$

Here, k is the discrete-time index and the operator σ is a ring automorphism on A. The inverse σ^{-1} of σ is the left-shift operator on A. The ring A with the ring automorphism σ is called a difference ring.

For any positive integer n, let $\mathbf{R}^n$ denote the vector space consisting of all n-element column vectors over the field $\mathbf{R}$. Then in terms of the above constructions, we have the following notion of a system.

(2.1) DEFINITION. An m-input p-output n-dimensional linear time-varying discrete-time system over A is a triple (F,G,H) of matrices over A where F is n x n , G is n x m , and H is p x n , together with the dynamical equations

$$x(k+1) = F(k)x(k) + G(k)u(k)$$
$$y(k) = H(k)x(k) \quad .$$

Here, $x(k) \in \mathbf{R}^n$ is the state at time k, $u(k) \in \mathbf{R}^n$ is the input at time k, and $y(k) \in \mathbf{R}^p$ is the output at time k.

For any positive integer n, let A^n denote the free A-module consisting of all n-element column vectors over A. Given an n x n matrix F over A, let S_F denote the operator on A^n defined by

(2.2)
$$S_F : A^n \longrightarrow A^n$$
$$v(k) \longmapsto F(\sigma v) = F(k)v(k-1), \text{ for all integers } k.$$

Note that $S_F(av) = F\sigma(av) = F(\sigma a)(\sigma v) = (\sigma a)S_F(v)$. The operator S_F is said to be a semilinear transformation (KAMEN and HAFEZ [1979]).

The system (F,G) over A is said to be reachable in N steps at all times (or N-step reachable at all times) if for any state x belonging to n-dimensional Euclidean space $\mathbf{R}^n$ and for any time k , there is a control sequence $\{u(k-N),u(k-N+1), \ldots , u(k-1)\}$ that drives the system from $x(k-N) = 0$ to $x(k) = x$. A system (F,G,H) over A is said to be observable in N steps at all times if and only if the system (F',H') is reachable in N steps at all times, where the prime denotes transposition.

Let $C_N(k)$ denote the n x mN reachability matrix defined by

$$C_N(k) = [G(k)\ (S_F G)(k)\ \ldots\ (S_F{}^{N-1}G)(k)] \quad .$$

It is well known (WEISS [1972]) that the system (F,G) is <u>N-step reachable at all times if and only if</u>

$$\text{rank } C_N(k) = n \quad , \quad \text{for all integers } k.$$

(2.3) <u>EXAMPLE</u>. Consider the system (F,G) over A given by $F(k) = \begin{bmatrix} 1 & 0 \\ od(k) & 1 \end{bmatrix}$ and $G(k) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ where od(k) = 0 if k is even and od(k) = 1 otherwise.

Then $C_3(k) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & od(k) & 1 \end{bmatrix}$ which has rank 2 for all integers k. Thus the system is reachable in 3 steps but not in 2 steps. Therefore, in discrete-time time-varying systems, the system (F,G) can be reachable in N steps at all times, where N > n. This situation may occur since we do not have a "Cayley-Hamilton" theorem in the time-varying case. []

Let the reachability gramian $W(k_0,k_1)$ for integers $k_1 > k_0$ be the symmetric nonnegative-definite matrix defined by

$$W(k_0,k_1) = \sum_{k=k_0}^{k_1-1} \Phi(k_1,k+1)G(k)G'(k)\Phi'(k_1,k+1)$$

where the prime denotes transposition and where $\Phi(k_1,k+1)$ is the transition matrix of the system given by (see KWAKERNAAK and SIVAN [1972])

$$\Phi(k,k_0) = \begin{cases} F(k-1)F(k-2) \; . \; . \; . \; F(k_0) & , \quad k > k_0+1 \\ I & , \quad k=k_0 \end{cases}$$

Then it can be shown that $W(k_0,k_1) = C_N(k_1-1)C_N'(k_1-1)$, where $N = k_1 - k_0$. Therefore, the system (F,G) is N-step reachable at all times if and only if there exists a fixed integer N such that $W(k_0,k_1)$ is nonsingular at all times $k_0 \in Z$ with $k_1 = k_0 + N$.

Let C(k) be any p x q matrix over A such that

$$\text{rank } C(k) \leq n \quad \text{for all } k \in Z \quad .$$

Then there exist a q x q matrix M(k) over A and an p x n matrix W(k) over A with rank M(k) = q and rank W(k) $\leq$ n for all integers k, and such that

$$C(k)M(K) = [W(k) \; : \; 0] \quad .$$

This is the discrete-time version of Dolezal's theorem (see FERRER [1984]).

Let us consider column vectors $v_1, v_2, \; . \; . \; . \; . \; , v_q$ belonging to A^n, where q is a positive integer. We say that v_q is an <u>A-linear</u>

<u>combination</u> of $v_1, v_2, \ldots, v_{q-1}$ if there exist $a_1, a_2, \ldots, a_{q-1} \in A$ such that $v_q = \sum_{i=1}^{q-1} a_i v_i$.

Again let (F,G) be an m-input N-step reachable system with reachability matrix $C_N(k)$. Assume that rank G(K) = m for at least one value of k. This does not result in any loss of generality. In particular, if rank G(k) < m for all integers k, by the discrete-time version of Dolezal's theorem, there is an m x m invertible matrix M(k) and an n x $\tilde{m}$ matrix $\tilde{G}(k)$ with $\tilde{m} < m$ such that

$$G(k)M(k) = [\tilde{G}(k) \mid 0]$$

with rank $\tilde{G}(k) = \tilde{m}$ for at least one value of k. Then, we can work with the pair $(F,\tilde{G})$, since if we let

$$M(k) = [M_1(k) \mid M_2(k)]$$

where $M_1(k)$ is an m x $\tilde{m}$ matrix and $M_2(k)$ is an m x $(m-\tilde{m})$, then the system (F,G) is equivalent to $(F,\tilde{G})$ via the input transformation $M_1(k)$.

With the columns of G denoted by $g_1, g_2, \ldots, g_m$, consider the array of vectors shown below:

$$\begin{array}{cccc}
g_1 & g_2 & \cdots & g_m \\
S_F g_1 & S_F g_2 & \cdots & S_F g_m \\
\vdots & \vdots & \ddots & \vdots \\
S_F^{N-1} g_1 & S_F^{N-1} g_2 & \cdots & S_F^{N-1} g_m
\end{array} .$$

By searching the rows of this array from left to right, we select those vectors that are not equal to an A-linear combination of previously selected vectors. Vectors that are A-linear combinations of previously selected vectors are discarded. In this selection process, it is easy to see that if $S_F{}^j g_i$ is discarded for some integers i,j (where $1 \leq i \leq m$ and $1 \leq j \leq N-1$), then $S_F{}^\ell g_i$ will also be discarded for $j \leq \ell \leq N-1$. Hence, the selection process will result in an array of the form

$$\begin{array}{cccc}
g_1 & g_2 & \cdots & g_m \\
S_F g_1 & S_F g_2 & \cdots & S_F g_m \\
\vdots & \vdots & & \vdots \\
S_F{}^{n_1-1} g_1 & S_F{}^{n_2-1} g_2 & & S_F{}^{n_m-1} g_m
\end{array}$$

There are no gaps in this array; that is, if $S_F{}^j g_i$ is in the array, then $S_F{}^r g_i$ is in the array for $0 \leq r \leq j$. Since the given system was assumed to be N-step reachable at all times, the vectors in the above array must generate A^n.

In this work, we shall refer to the integers $n_1, n_2, \ldots, n_m$ in the above array as <u>the generalized controllability indices</u> of the system. If the system (F,G) is time invariant, the n_i are the usual controllability indices of the system. Letting $q = n_1 + n_2 + \ldots + n_m$, we have that $N \leq q \leq mN$. If $N = n$ (i.e., the system is n-step reachable), and $q = n$, we shall say that the system (F,G) has <u>a fixed</u>

basis. Note that any reachable time-invariant system has a fixed basis. If the system (F,G) has a fixed basis, a reodering of the elements of the above array corresponds to a fixed lexicographic basis as derived in the work of WOLOVICH [1968].

A system (F,G) is said to be index invariant (MORSE and SILVERMAN [1973]) if, for each $i = 1, 2, \ldots, n$, rank $C_i(k) = q_i$ = constant, for all integers k, and $q_n = n$ where $C_i(k) = [G(k)\ (S_F G)(k) \ldots (S_F^{i-1} G)(k)]$.

Any index-invariant system and any system having a fixed basis are n-step reachable systems. But the converse is not true. Any system with a fixed basis is index invariant. But an index-invariant system does not always have a fixed basis. Therefore, the existence of a fixed basis is a very strong condition in the time-varying case.

(2.4) EXAMPLE. Let $F(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $G(k) = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$. Then $C_2(k) = \begin{bmatrix} k & 0 & k-1 & 0 \\ 0 & k & 0 & k-1 \end{bmatrix}$ which has rank 2 for all integers k. Thus the system is reachable in 2 steps, but (F,G) is not index invariant and does not have a fixed basis. []

(2.5) EXAMPLE. Let $F(k) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and $G(k) = \begin{bmatrix} k & k+1 \\ 0 & 0 \end{bmatrix}$. Then $C_2(k) = \begin{bmatrix} k & k+1 & 0 & 0 \\ 0 & 0 & k-1 & k \end{bmatrix}$ which has rank 2 for all integers k. Thus the system

is reachable in 2 steps. Since rank $G(k) = 1$ for all k, the system is also index invariant. But the system does not have a fixed basis. []

From the above discussion, we see that N-step reachable systems are the largest class among the other classes of systems mentioned above, including n-step reachable systems, index-invariant systems and systems having a fixed basis.

Let $A[z]$ denote the set of all finite sums of the form $\sum_{i\geq 0} z^i a_i$ in the indeterminate z with coefficients $a_i \varepsilon A$ written on the right. With the usual addition and multiplication defined in terms of

$$
\begin{aligned}
z^i z^j &= z^{i+j} \quad , \quad i,j \varepsilon Z, \quad i,j > 0 \\
az &= z(\sigma a) \quad , \quad a \varepsilon A \quad ,
\end{aligned}
\tag{2.6}
$$

$A[z]$ is a noncommutative ring called a (right) skew polynomial ring or a polynomial difference ring.

A semilinear transformation S_F defined by (2.2) induces a right $A[z]$-module structure on A^n with addition and scalar multiplication given by

$$
\begin{aligned}
(v + w)(k) &= v(k) + w(k) \\
v(\sum_i z^i a_i) &= \sum_i a_i S_F{}^i(v)
\end{aligned}
\tag{2.7}
$$

where $v,w \in A^n$, $a_i \in A$ and

$$(S_F^i v)(k) = F(k)(S_F^{i-1}v)(k-1) \text{ for } i = 1, 2, \ldots$$
$$(S_F^0 v)(k) = v(k).$$

Let $p(z)$ be an element of $A[z]$ given by $\sum_{i\geq 0} z^i a_i$. Then the degree of $p(z)$, denoted by deg $p(z)$, is the maximum value of i such that $a_i \neq 0$. If deg $p(z) = n$ and $a_n = 1$, then $p(z)$ is said to be monic. Let $A((z^{-1}))$ denote the set of all formal Laurent series of the form

$$\sum_{i=-N}^{\infty} z^{-i} a_i \quad , \quad a_i \in A \quad , \quad N \in Z \quad .$$

With the usual addition, and with multiplication defined by (2.6) and

$$az^{-1} = z^{-1}(\sigma^{-1}a) \quad , \quad a \in A \quad ,$$

$A((z^{-1}))$ is a noncommutative ring with identity, called the skew ring of formal Laurent series over A. The skew ring of formal power series $A[[z^{-1}]]$ defined by

$$A[[z^{-1}]] = \{a \text{ in } A((z^{-1})) : a = \sum_{i=0}^{\infty} z^{-i} a_i\}$$

is a subring of $A((z^{-1}))$. A polynomial $p(z) \in A[z]$ is said to be invertible if and only if there exists a $q(z)$ in $A((z^{-1}))$ such that $p(z)q(z) = q(z)p(z) = 1$.

A single-input n-dimensional system (F,g) is said to be N-cyclic with $N > n$ if and only if for any $v \in A^n$ there exist $a_0, a_1, \ldots, a_{N-1} \in A$ such that

$$v = \sum_{i=0}^{N-1} a_i S_F{}^i g \quad .$$

From the multiplication of the A[z]-module structure on A^n defined by (2.7), we can express the above equation as $v = g\pi(z)$, where $\pi(z)$ is the polynomial in A[z] given by

$$\pi(z) = \sum_{i=0}^{N-1} z^i a_i \quad .$$

We note that the notion of N-cyclicity is equivalent to N-step reachability.

Given $g \in A^n$, let Ann g denote the annihilator of g defined by Ann g = $\{p(z) \in A[z] : gp(z) = 0\}$. Then Ann g is a right ideal of the ring A[z]. KAMEN and HAFEZ [1979] have shown that <u>(F,g) is n-cyclic (n-step reachable) if and only if Ann g = a(z)A[z]</u>, where a(z) is a monic polynomial of degree n. This monic polynomial a(z) is called the <u>order of g</u>. Let $a(z) = z^n + \sum_{i=0}^{n-1} z^i a_i$. From (2.7), we see that

$$ga(z) = S_F{}^n g + \sum_{i=0}^{n-1} a_i S_F{}^i g \quad .$$

Therefore, $[a_0\ a_1\ \ldots\ a_{n-1}]' = -C_n{}^{-1}(S_F{}^n g)$, where the prime denotes transposition (see KAMEN and HAFEZ [1979]).

From the results of KAMEN and HAFEZ [1979], if a single-input system (F,g) is n-step reachable, then there exists a coordinate transformation $\bar{x}(k) = T(k)x(k)$ such that

$$\bar{F}(k) = T(k+1)F(k)T^{-1}(k) \quad ,$$

(2.8)

$$\bar{g}(k) = T(k+1)g(k)$$

where

$$F(k) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & & \ddots & \\ \vdots & \cdots & & 1 \\ -a_0(k) & -a_1(k) & \cdots & -a_{n-1}(k) \end{bmatrix} \quad , \quad \bar{g}(k) = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$$

and $[a_0 \; a_1 \; \cdots \; a_{n-1}]' = -C_n^{-1}(S_F{}^n g)$.

If we let $\bar{C}_n(k) = [\bar{g}(k) \; (S_{\bar{F}}\bar{g})(k) \; \cdots \; (S_{\bar{F}}{}^{n-1}\bar{g})(k)]$, then T(k) is given by

(2.9) $$T(k) = \bar{C}_n(k-1)C_n^{-1}(k-1) \quad .$$

The form (2.8) is called the <u>control canonical form</u> which is the discrete-time counterpart to SILVERMAN [1966]'s phase variable form in the continuous-time case.

CHAPTER THREE
AUGMENTATION OF THE SYSTEM STATE

In this chapter, we shall present a constructive method for augmenting a N-step reachable time-varying system so that the augmented system has a control canonical form.

If a given n-dimensional discrete-time time-varying system is N-step reachable with $N > n$, we will first show that it is possible to turn the given system into a system having a fixed basis by simply augmenting the system state. This fixed basis leads to the construction of a control canonical form for the augmented system. Then, via the control canonical form, we can obtain a feedback gain matrix which results in the closed-loop system matrix of the augmented system being algebraically equivalent to a constant matrix with arbitrary assignable eigenvalues. Therefore, by using dynamic state feedback, which consists of the augmentation plus the feedback gain matrix, we have an assignability type result for N-step reachable time-varying systems. This approach to assignability yields a controller having a much lower order in general than the controller derived by KAMEN et al. [1985]. More details about the construction of feedback control systems will be given in Chapter Four.

Let (F,G) denote the m-input n-dimensional time-varying system given by the state equation

$$x(k+1) = F(k)x(k) + G(k)u(k) \quad . \tag{3.1}$$

Given a positive integer r, let D(k), B(k), E(k) be r x n, r x r, r x m time-varying matrices, and consider the system

$$w(k+1) = D(k)x(k) + B(k)w(k) + E(k)u(k) \quad . \tag{3.2}$$

The system given by (3.2) is an augmentation of the given system (3.1). The r x 1 vector w(k) is the state vector of the augmentation.

Combining the system equation (3.1) and the augmentation (3.2), we obtain the following state equation for the augmented system

$$\begin{bmatrix} x(k+1) \\ \\ w(k+1) \end{bmatrix} = \begin{bmatrix} F(k) & 0 \\ \\ D(k) & B(k) \end{bmatrix} \begin{bmatrix} x(k) \\ \\ w(k) \end{bmatrix} + \begin{bmatrix} G(k) \\ \\ E(k) \end{bmatrix} u(k) \tag{3.3}$$

Figure 3.1 shows the block diagram of the augmented system.

In the following, we let $\hat{F}(k)$, $\hat{G}(k)$ denote the matrices defined by

$$\hat{F}(k) = \begin{bmatrix} F(k) & 0 \\ D(k) & B(k) \end{bmatrix} , \quad \hat{G}(k) = \begin{bmatrix} G(k) \\ E(k) \end{bmatrix}$$

To derive the main results, we need the following proposition.

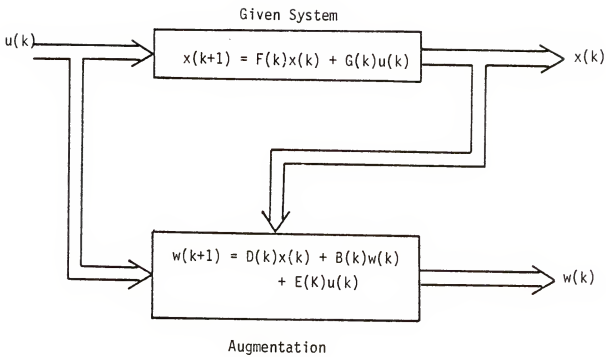

Fig. 3.1 The block diagram of the augmented system

(3.4) PROPOSITION. Let $X(k)$ be an $n \times \ell$ matrix over A with $\ell > n$. Suppose that rank $X(k) = n$ for all integers k. Then there always exists an $(\ell-n) \times \ell$ matrix $Y(k)$ over A such that

$$\operatorname{rank}\begin{bmatrix} X(k) \\ Y(k) \end{bmatrix} = \ell \quad \text{for all integers } k.$$

PROOF. Since rank $X(k) = n$ for all integers k, by the discrete-time version of Dolezal's theorem , there exist an $\ell \times \ell$ invertible matrix $M(k)$ and an $n \times n$ invertible matrix $W(k)$ such that

$$X(k)M(k) = [W(k) \mid 0] \quad .$$

Let $Y(k) = [0 \; W_1(k)]M^{-1}(k)$, where $W_1(k)$ is an arbitrary invertible $(\ell-n) \times (\ell-n)$ matrix. Then we have

$$\begin{bmatrix} X(k) \\ Y(k) \end{bmatrix} = \begin{bmatrix} W(k) & 0 \\ 0 & W_1(k) \end{bmatrix} M^{-1}(k)$$

Since $\begin{bmatrix} W(k) & 0 \\ 0 & W_1(k) \end{bmatrix}$ is invertible , $\begin{matrix} X(k) \\ Y(K) \end{matrix}$ is also invertible. Therefore, there exists a $Y(k)$ such that $\begin{bmatrix} X(k) \\ Y(K) \end{bmatrix}$ is invertible. We note that $Y(k)$ is not unique.

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Now we have the following results in the single-input case.

(3.5) THEOREM. Suppose that (F,g) is N-step reachable, where $N > n$. Then there always exist an $(N-n) \times n$ matrix $D(k)$, an $(N-n) \times (N-n)$ matrix $B(k)$ and an $(N-n) \times 1$ vector $e(k)$ such that the augmented N-dimensional system $(\hat{F},\hat{g})$ is N-step reachable, where

$$\hat{F} = \begin{bmatrix} F & 0 \\ D & B \end{bmatrix}, \quad \hat{g} = \begin{bmatrix} g \\ e \end{bmatrix}.$$

PROOF. Since (F,g) is N-step reachable, rank $C_N(k) = n$ for all integers k, where $C_N(k) = [g \; S_F g \; . \; . \; . \; S_F^{N-1} g]$. From Proposition 3.4, it is always possible to choose $(N-n)$-element column vectors $e_0(k)$, $e_1(k)$, $. \; . \; . \; , \; e_{N-1}(k)$ such that

$$\text{rank} \begin{bmatrix} g(k) & (S_F g)(k) & . \; . \; . & (S_F{}^{N-1} g)(k) \\ e_0(k) & e_1(k) & . \; . \; . & e_{N-1}(k) \end{bmatrix} = N \quad \text{for all integers } k.$$

Let $\hat{C}(k)$ denote the $N \times N$ matrix

$$(3.6) \qquad \hat{C}(k) = \begin{bmatrix} g(k) & (S_F g)(k) & . \; . \; . & (S_F{}^{N-1} g)(k) \\ e_0(k) & e_1(k) & . \; . \; . & e_{N-1}(k) \end{bmatrix}$$

Define

$$e(k) = e_0(k) \quad , \quad [D(k)\ B(k)] = [e_1(k)\ .\ .\ .\ e_N(k)](\sigma\hat{C})^{-1}(k),$$

where $e_N(k)$ is an arbitrary (N-n)-element column vector, and let

$$\hat{F}(k) = \begin{bmatrix} F(k) & 0 \\ D(k) & B(k) \end{bmatrix} \quad , \quad \hat{g}(k) = \begin{bmatrix} g(k) \\ e(k) \end{bmatrix} \quad .$$

By applying the operator $S_{\hat{F}}$ to both sides of (3.6), we have

$$(S_{\hat{F}}\hat{C})(k) = \begin{bmatrix} (S_F g)(k) & . \ . \ . & (S_F^{N-1} g)(k) & (S_F^{N} g)(k) \\ e_1(k) & . \ . \ . & e_{N-1}(k) & e_N(k) \end{bmatrix}$$

Therefore, $\hat{C}(k)$ is the reachability matrix of the augmented system $(\hat{F},\hat{g})$, and thus the theorem is proved. []

Since the N-dimensional augmented system is N-step reachable , we can apply the result of KAMEN and HAFEZ [1979] to construct a control canonical form for the augmented system. More precisely, we have the following result.

(3.7) <u>THEOREM</u>. If (F,g) is N-step reachable, then there is an N-dimensional augmented system $(\hat{F},\hat{g})$ and an N x N invertible matrix $T(k)$ such that $T(k+1)\hat{F}(k)T^{-1}(k)$ and $T(K+1)\hat{g}(k)$ are in control canonical form.

PROOF. Combining (2.8), (2.9) and Theorem 3.5, the theorem follows. []

(3.8) REMARK. The control canonical form $(\bar{F},\bar{g})$, where $\bar{F} = T(k+1)\hat{F}(k)T(k)$ and $\bar{g} = T(k+1)\ \hat{g}(k)$, is called a generalized control canonical form in the single-input time-varying case. []

We illustrate the above results by the following example.

(3.9) EXAMPLE. Consider the single-input time-varying system (F,g) given by

$$F(k) = \begin{bmatrix} 1 & 0 \\ od(k) & 1 \end{bmatrix}, \quad g(k) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

where $od(k) = \begin{cases} 1, & k \text{ is odd} \\ 0, & k \text{ is even} \end{cases}$.

As in Example 2.2, the reachability matrix is

$$C_3(k) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & od(k) & 1 \end{bmatrix}$$

and this system is 3-step reachable.

Let $\hat{C}_3(k) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & od(k) & 1 \\ 1 & 0 & 1 \end{bmatrix}$. Then, rank $\hat{C}_3(k) = 3$ for all integers k.

Define $e(k) = 1$, $e_3(k) = 0$, $[D(k)\ B(k)] = [0\ 1\ 0](\sigma\hat{C}_3^{-1}(k)) = [1\ 0\ -1]$.

Then we have $\hat{F}(k) = \begin{bmatrix} 1 & 0 & 0 \\ od(k) & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$, $\hat{g}(k) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Since $\hat{C}^{-1}(S_{\hat{F}}{}^{3}\hat{g}) = [1\ \ -1\ \ -1]'$, the control canonical form of the augmented system is

$$\bar{F}(k) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}, \qquad \bar{g}(k) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and the coordinate transformation is $T(k+1) = \begin{bmatrix} -od(k) & 1 & od(k) \\ ev(k) & 1 & -ev(k) \\ ev(k) & 1 & od(k) \end{bmatrix}$

where $ev(k) = \begin{cases} 1 & , k \text{ is even} \\ 0 & , k \text{ is odd.} \end{cases}$ []

Now let us consider the m-input n-dimensional time-varying system (F,G). In the following, we assume that rank $G(k) = m$ for at least one

integer value of k. If rank $G(k) < m$ for all integers k, then the augmentation will have the structure shown in Fig. 3.2, where $\tilde{u}$ is the input of the system $(F,\tilde{G})$ and $\tilde{G}$, $M_1(k)$ are defined in Chapter Two.

(3.10) THEOREM. Suppose that (F,G) is N-step reachable and rank $G(k)=m$ for at least one integer value of k. Then for some integer q where $N \leq q \leq mN$ there exists a (q-n)-dimensional augmentation

$$(3.11) \qquad w(k+1) = D(k)x(k) + B(k)w(k) + E(k)u(k)$$

such that the q-dimensional augmented system $(\hat{F},\hat{G})$ is q-step reachable with a fixed basis.

PROOF. We would like to prove the above theorem step by step in a constructive way.

STEP I. Using the selection process described in Chapter Two, determine the generalized controllability indices of the system, $n_1, n_2, \ldots, n_m$. For $1 \leq i \leq m$, we can then express $S_F^{n_i} g_i$ as an A-linear combination of previously selected vectors; that is

$$(3.12) \qquad S_F^{n_i} g_i = - \sum_{j=0}^{n_i} \sum_{\ell=1}^{m} r_j^{\ell i} (S_F^j g_\ell) \quad , \quad i=1, 2, \ldots, m$$

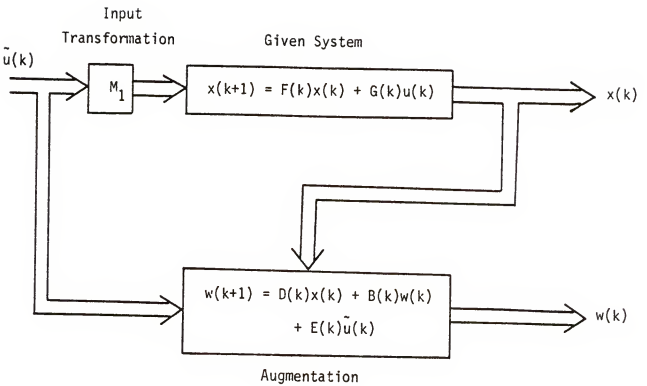

Fig. 3.2 Augmented system when rank G(k) < m

where $r_j^{\ell i} \in A$ with

$$\begin{cases} r_{n_m}^{\ell i} = 0 & \text{for} \quad \ell = i, i+1, \ldots, m \\ \\ r_{n_\ell}^{\ell i} = r_{n_\ell+1}^{\ell i} = \cdots = r_{n_i}^{\ell i} = 0 & \text{for} \quad \ell = 1, 2, \ldots, m \end{cases}$$

STEP II. Choose q vectors $e_{ij}(k)$, $1 \leq j \leq m$, $0 \leq i \leq n_j - 1$, where $e_{ij}(k)$ are $(q-n) \times 1$ vectors, such that

$$\text{rank} \begin{bmatrix} g_1 & S_F g_1 & \cdots & S_F^{n_1-1} g_1 & g_2 & S_F g_2 & \cdots & g_m & \cdots & S_F^{n_m-1} g_m \\ \\ e_{01} & e_{11} & \cdots & e_{(n_1-1)1} & e_{02} & e_{12} & \cdots & e_{0m} & \cdots & e_{(n_m-1)m} \end{bmatrix} = q$$

at all times.

This is always possible from Proposition 3.4. Let $\tilde{C}_q(k)$ denote the $q \times q$ matrix

$$(3.13) \quad \tilde{C}_q = \begin{bmatrix} g_1 & S_F g_1 & \cdots & S_F^{n_1-1} g_1 & g_2 & \cdots & g_m & \cdots & S_F^{n_m-1} g_m \\ \\ e_{01} & e_{11} & \cdots & e_{(n_1-1)1} & e_{02} & \cdots & e_{0m} & \cdots & e_{(n_m-1)m} \end{bmatrix}$$

Using the coefficients $r_j^{\ell i}$ in (3.12), calculate $e_{n_i i}$, $i = 1,2, \ldots, m$ such that

$$(3.14) \quad e_{n_i i} = - \sum_{j=0}^{n_i} \sum_{\ell=1}^{m} r_j^{\ell i} e_{j\ell} \quad , \quad i = 1, 2, \ldots, m \quad .$$

STEP III. Define $E(k) = [e_{01}(k)\ e_{02}(k)\ \ldots\ e_{0m}(k)]$ and

$$[D(k)\ B(k)] = X(k)(\sigma\tilde{C}q)^{-1}(k)$$

where $X(k) = [e_{11} \ldots e_{n_1 1}\ e_{12} \ldots e_{1m} \ldots e_{n_m m}]$.

Let $\hat{F} = \begin{bmatrix} F & 0 \\ D & B \end{bmatrix}$ and $\hat{G} = \begin{bmatrix} G \\ E \end{bmatrix}$. Apply the operator $S_{\hat{F}}$ to both sides of (3.13). Then we have

$$S_{\hat{F}}\tilde{C}q = \begin{bmatrix} S_F g_1 & \ldots & S_F^{n_1} g_1 & S_F g_2 & \ldots & S_F g_m & \ldots & S_F^{n_m} g_m \\ e_{11} & \ldots & e_{n_1 1} & e_{12} & \ldots & e_{1m} & \ldots & e_{n_m m} \end{bmatrix}$$

From (3.12) and (3.14), we see that

$$S_{\hat{F}}^{n_i} \hat{g}_i = - \sum_{j=0}^{n_i} \sum_{\ell=1}^{m} r_j^{\ell i} (S_{\hat{F}}^{j} \hat{g}_\ell) \quad , \quad i = 1, 2, \ldots, m$$

where $\hat{g}_i$ is the ith column vector of $\hat{G}$ and $\hat{g}_i$, $S_{\hat{F}}{}^{j}\hat{g}_\ell$ are given by

$$\hat{g}_i = \begin{bmatrix} g_i \\ e_{0i} \end{bmatrix} , \qquad S_{\hat{F}}{}^{j}\hat{g}_\ell = \begin{bmatrix} S_F{}^{j} g_\ell \\ e_{j\ell} \end{bmatrix}.$$

Therefore, $\tilde{C}_q$ will be a part of the reachability matrix $\hat{C}_N$ for the augmented system $(\hat{F},\hat{G})$, where $\hat{C}_N$ is given by

$$\hat{C}_N = [\hat{G} \; S_{\hat{F}}\hat{G} \; . \; . \; . \; S_{\hat{F}}{}^{q-1}\hat{G}] \quad .$$

Then the theorem follows. []

We should note that by equation (3.14), the controllability indices of the augmented system are the same as the indices of the given system which are determined in Step I of the proof of Theorem 3.10.

We now define a control canonical form for the time-varying system. Suppose that the m-input n-dimensional time-varying system (F,G) has a coordinate transformation $\bar{x}(k) = T(k)x(k)$ such that $\bar{F}(k) = T(k+1)F(k)T^{-1}(k)$ and $\bar{G}(k) = T(k+1)G(k)$ are in the following form

$$\bar{F} = \begin{bmatrix} F_{11} & F_{12} & . \; . \; . & F_{1m} \\ F_{21} & & . \; . \; . & F_{2m} \\ & & . \; . \; . & \\ F_{m1} & & . \; . \; . & F_{mm} \end{bmatrix} , \quad \bar{G} = \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_m \end{bmatrix} \tag{3.15}$$

where $$F_{ii} = \left[\begin{array}{c|c} 0_{(n_i-1)\times 1} & I_{n_i-1} \\ \hline X & X \;.\;.\;.\; X \end{array}\right] \quad \underset{i\neq j}{F_{ij}} = \left[\begin{array}{c} 0_{(n_i-1)\times n_j} \\ \hline X \;.\;.\;.\; X \end{array}\right]$$

$$G_i = \left[\begin{array}{c|c} 0_{n_i \times (i-1)} & 0_{(n_i-1)\times(n-i+1)} \\ \hline & 1 \quad X \;.\;.\;.\; X \end{array}\right]$$

and X's are coefficients in A.

Then the form (3.15) is called <u>the control canonical form</u> of the multivariable time-varying system. For the control canonical form of the time-invariant case, the reader may refer to POPOV [1972] ,KAILATH [1980], and CHAN and WANG [1978].

Since the augmented system constructed in the proof of Theorem 3.10 has a fixed basis, we can construct the control canonical form for the augmented system.

(3.16) <u>THEOREM</u>. Let $(\hat{F},\hat{G})$ be the augmented system constructed in the proof of Theorem 3.10. Then there is a coordinate transformation $\bar{x}(k) = T(k)x(k)$ such that $\bar{F}(k) = T(k+1)\hat{F}(k)T^{-1}(k)$ and $\bar{G}(k) = T(k+1)\hat{G}(k)$ are in control canonical form.

<u>PROOF</u>. This proof is a generalization of the proof in the time-invariant case. The reader may refer to CHAN and WANG [1978].

Let us denote $\hat{G} = [\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_m]$. Using the coefficients in equation (3.12), define the following q-element column vectors

$$(3.17) \qquad x_t^i = S_{\hat{F}}^t \hat{g}_i + \sum_{j=0}^{t} \sum_{\ell=1}^{m} (\sigma^{-(n_i-t)} r_{n_i-j}^{\ell i})(S_{\hat{F}}^{t-j} \hat{g}_\ell)$$

$$i=1,2, \ldots m \; ; \; t = 0,1, \ldots, n_i \quad .$$

Let T_x be the q x q matrix defined by

$$(3.18) \quad T_x = [x_{n_1-1}^1 \ldots x_0^1 \; ; \; x_{n_2-1}^2 \ldots x_0^2 \; ; \; \ldots \; ; \; x_{n_m-1}^m \ldots x_0^m].$$

Then it can be shown that $v'T_x = 0$ if and only if $v = 0$. Hence, T_x is invertible (see POPOV [1972]). Now, we must show that T_x is a coordinate transformation such that

$$(3.19) \qquad T_x \bar{F} = \hat{F} \sigma T_x$$

$$(3.20) \qquad T_x \bar{G} = \hat{G}$$

where $\bar{F}$ and $\bar{G}$ are in control canonical form. From (3.17) and the coefficients in equation (3.12), we have

$$x_0^i = \hat{g}_i + \sum_{\ell=1}^{i-1} (\sigma^{-n_i} r_{n_i}^{\ell i}) \hat{g}_i \quad , \quad i = 1, 2, \ldots, m \quad .$$

If we write these equations in a matrix format, we have

$$(3.21)\quad [x_0^1 \ldots x_0^m] = [\hat{g}_1 \ldots \hat{g}_m] \begin{bmatrix} 1 & \sigma^{-n_2} r_{n_2}^{12} & \ldots & \sigma^{-n_m} r_{n_m}^{1m} \\ 0 & 1 & & \\ \vdots & & \ddots & \vdots \\ & & & \sigma^{-n_m} r_{n_m}^{(m-1)m} \\ 0 & & \ldots & 1 \end{bmatrix}$$

$$\text{Let} \quad Q = \begin{bmatrix} 1 & \sigma^{-n_2} r_{n_2}^{12} & \ldots & \sigma^{-n_m} r_{n_m}^{1m} \\ 0 & 1 & & \vdots \\ & & \ddots & \\ \vdots & & & \\ 0 & & & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & b^{12} & \ldots & b^{1m} \\ 0 & 1 & \ldots & b^{2m} \\ & & & \vdots \\ & & \ddots & b^{(m-1)m} \\ & & & 1 \end{bmatrix}$$

Then we have $T_x\bar{G} = \hat{G}$ by inserting zero rows in Q, where

$$(3.22)\quad \bar{G} = \begin{bmatrix} & 0_{(n_1-1) \times m} & & \\ 1 & b^{12} & \ldots & b^{1m} \\ \hline & 0_{(n_2-1) \times m} & & \\ 0 \;\; 1 & b^{23} & \ldots & b^{2m} \\ \hline & \vdots & & \vdots \\ \hline & 0_{(n_m-1) \times m} & & \\ 0 & & \ldots & 1 \end{bmatrix}$$

To prove (3.19) , we apply $S_{\hat{F}}$ to both sides of (3.17). Then

$$(3.23)\quad S_{\hat{F}}\, x_t^i = S_{\hat{F}}^{t+1}\hat{g}_i + \sum_{j=0}^{t}\sum_{\ell=1}^{m}(\sigma^{-(n_i-t-1)}\, r_{n_i-j}^{\ell i})(S_{\hat{F}}^{t-j+1}\hat{g}_\ell)$$

$$= x_{t+1}^i - \sum_{\ell=1}^{m}(\sigma^{-(n_i-t-1)}\, r_{n_i-t-1}^{\ell i})\hat{g}_\ell$$

$$= x_{t+1}^i - [\hat{g}_1 \;.\;.\;.\; \hat{g}_m](Q^{-1})(Q)\begin{bmatrix}\sigma^{-(n_i-t-1)}\, r_{n_i-t-1}^{1i}\\ \vdots \\ \sigma^{-(n_i-t-1)}\, r_{n_i-t-1}^{mi}\end{bmatrix}$$

$$= x_{t+1}^i - [x_0^1 \;.\;.\;.\; x_0^m]\begin{bmatrix}a_{n_i-(t+1)}^{1i}\\ \vdots \\ a_{n_i-(t+1)}^{mi}\end{bmatrix}$$

$$i = 1, 2, \ldots, m\ ;\ t = 0, 1, \ldots, n_i-1,$$

where

$$\begin{bmatrix}a_{n_i-t-1}^{1i}\\ \vdots \\ a_{n_i-t-1}^{mi}\end{bmatrix} = (Q)\begin{bmatrix}\sigma^{-(n_i-t-1)}\, r_{n_i-t-1}^{1i}\\ \vdots \\ \sigma^{-(n_i-t-1)}\, r_{n_i-t-1}^{mi}\end{bmatrix}$$

By inserting zeros in (3.23) and writing these equations in a matrix format, we have

$$S_{\hat{F}} T_x = T_x F$$

where

$$(3.24)\quad F_{ii} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ & & & \ddots & \\ \vdots & \vdots & & & 1 \\ -a_0^{ii} & -a_1^{ii} & & \cdots & -a_{n_i-1}^{ii} \end{bmatrix}$$

$$i=1,2,\ \ldots\ ,\ m$$

$$F_{ij} = \begin{bmatrix} & & \mathbf{0}_{(n_i-1)\times(m-i\times 1)} & & \\ -a_0^{ij} & -a_1^{ij} & \cdots & & -a_{n_j-1}^{ij} \end{bmatrix}$$

$$i,\ j,\ =\ 1,\ 2,\ \ldots\ ,\ m \quad , \quad i \neq j.$$

Letting $T(k) = T_x^{-1}(k-1)$, we have $T(k)\bar{F}(k) = T(k+1)\hat{F}(k)$ and $\bar{G}(k) = T(k+1)\hat{G}(k)$.

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The control canonical form $(\bar{F},\bar{G})$ for the augmented system $(\hat{F},\hat{G})$ is called <u>a generalized control canonical form</u> in the multivariable time-varying case. We now illustrate the above results by the following example.

(3.25) <u>EXAMPLE</u>. Consider the system (F,G) given by

$$F(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad G(k) = \begin{bmatrix} 1 & 1 \\ 0 & od(k) \end{bmatrix}$$

where $od(k) = \begin{cases} 1 \ , \ k \text{ is odd} \\ 0 \ , \ k \text{ is even} \end{cases}$, $ev(k) = \begin{cases} 1 \ , \ k \text{ is even} \\ 0 \ , \ k \text{ is odd} \end{cases}$.

Then $C_2(k)$ is given by

$$C_2(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & od(k) & 0 & ev(k) \end{bmatrix}$$

which has rank 2 for all integers k. Thus the system is reachable in 2 steps. But $C_2(k)$ does not have a fixed basis. Now, let us construct the augmented system.

<u>STEP I</u>. We select g_1, g_2 and $S_F g_2$ from $C_2(k)$ and express $S_F{}^2 g_2$ and $S_F g_1$ as A-linear combinations of these previously selected vectors. Then

$$(S_F g_1)(k) = g_1(k)$$
$$(S_F{}^2 g_2)(k) = g_2(k)$$

The generalized controllability indices are $n_1 = 1$, $n_2 = 2$ and the dimension of the augmented system is $q = 3$.

<u>STEP II</u>. If we choose $e_{01} = 0$, $e_{02} = -1$, $e_{11} = 1$,

$$\tilde{C}_3(k) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & od(k) & ev(k) \\ 0 & -1 & 1 \end{bmatrix}$$

and rank $\tilde{C}_3 = 3$ at all times. Now, determine e_{11} and e_{22} from Step I and e_{01}, e_{02}, e_{11}. Then we have

$$e_{11} = 0$$

$$e_{22} = -1$$

<u>STEP III</u>. Define $E(k) = [0 \quad -1]$ and

$$\begin{aligned} [\, D \quad B \,] &= X(k)(\sigma\tilde{C}(k))^{-1} \\ &= [\, 0 \quad 1 \quad -1 \,] \begin{bmatrix} 1 & -2 & od(k)-ev(k) \\ 0 & 1 & -od(k) \\ 0 & 1 & ev(k) \end{bmatrix} \\ &= [\, 0 \quad 0 \quad -1 \,]. \end{aligned}$$

Hence, the augmented system $(\hat{F},\hat{G})$ will be

$$\hat{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \hat{G} = \begin{bmatrix} 1 & 1 \\ 0 & od(k) \\ 0 & -1 \end{bmatrix}$$

Now, let us construct a generalized control canonical form. From Step II,

$$(S_{\hat{F}}\hat{g}_1) = \hat{g}_1 \quad ,$$

$$(S^2_{\hat{F}}\hat{g}_2) = \hat{g}_2 \quad .$$

Define $\quad x_0^1 = \hat{g}_1 \quad , \quad x_1^1 = S_{\hat{F}}\hat{g}_2 \quad , \quad x_0^2 = \hat{g}_2 \quad ,$

and let $\quad T_x = [x_0^1 \;\; x_1^2 \;\; x_0^2]$. Then we have

$$S_{\hat{F}}T_x = [x_0^1 \;\; x_0^1 \;\; x_1^2] = T_x\bar{F}$$

$$\hat{G} = T_x\bar{G}$$

Therefore, the generalized control canonical form is

$$\bar{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \bar{G} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad []$$

In the following chapter, we shall construct a feedback gain matrix for the generalized control canonical form which results in the closed-loop system matrix being algebraically equivalent to a constant matrix with arbitrary assignable eigenvalues. Later, the augmentation approach will be applied to the design of output feedback controllers.

CHAPTER FOUR
FEEDBACK CONTROL

This chapter is divided into two parts. In the first part, we shall show how a feedback gain matrix can be constructed for augmented N-step reachable time-varying systems using the generalized control canonical form, resulting in assignability of the closed-loop system dynamics. If an n-dimensional time-varying system is N-step reachable with $N > n$, then it is possible to assign the coefficients of the closed-loop characteristic polynomial by employing a dynamic controller consisting of the feedback gain matrix and the augmentation. The system matrix of the resulting closed-loop system is algebraically equivalent to a constant matrix with arbitrary assignable eigenvalues. This assignability result for N-step reachable time-varying systems was first obtained by POOLLA [1984] and KAMEN et al. [1985]. But, we believe that our scheme for finding a controller is much more flexible, and our controller has a much lower order in general than the controller derived by KAMEN et al. [1985].

In the second part of this chapter, the augmentation approach is applied to design output feedback controllers. In this case, we can also get the coefficient assignability result. Later, the separation property of the observer-controller design procedure will be proved.

The augmented system constructed in Chapter Three for the N-step reachable time-varying system (F,G) is given by the following state equation

$$\hat{x}(k+1) = \hat{F}(k)\hat{x}(k) + \hat{G}(k)u(k) \tag{4.1}$$

where $\hat{x}(k) = \begin{matrix} x(k) \\ w(k) \end{matrix}$, $\hat{F}(k)$, $\hat{G}(k)$ are q x q , q x m matrices such that

$$\hat{F} = \begin{bmatrix} F & 0 \\ D & B \end{bmatrix} , \quad \hat{G} = \begin{bmatrix} G \\ E \end{bmatrix}$$

The problem is to find a nonsingular m x m matrix P(k) and an m x q gain matrix $\hat{L}(k)$ such that

$$u(k) = P(k)v(k) - \hat{L}(k)\hat{x}(k) \tag{4.2}$$

makes the closed-loop state equation

$$\hat{x}(k+1) = (\hat{F}(k) - \hat{G}(k)\hat{L}(k))\hat{x}(k) + \hat{G}(k)P(k)v(k) \tag{4.3}$$

have the preassigned characteristic polynomial where v(k) is a reference input.

From Theorem 3.16, there exists a coordinate transformation $\bar{x}(k) = T(k)\hat{x}(k)$ such that under this transformation, the state equation (4.3) becomes

$$\bar{x}(k+1) = (\bar{F}(k) - \bar{G}(k)\bar{L}(k))\bar{x}(k) + \bar{G}(k)P(k)v(k) \tag{4.4}$$

where

$$\bar{F}(k) = T(k+1)\hat{F}(k)T^{-1}(k) \quad , \quad \bar{G}(k) = T(k+1)\hat{G}(k)$$
$$\text{and } \bar{L}(k) = \hat{L}(k)T^{-1}(k) \quad .$$

The matrices $\bar{F},\bar{G}$ have the generalized control canonical form (3.15).

Let us determine the matrix $P(k)$ such that

$$(4.5) \qquad \bar{G}P = \text{block diag } \{[0 \ . \ . \ . \ 0 \ 1]', \ n_i \times 1 \ , \ i=1, \ . \ . \ . \ , \ m \}$$

where the above notation denotes the matrix whose diagonal elements are $n_i \times 1$ column vector $[0 \ . \ . \ . \ 0 \ 1]'$ for $1 \leq i \leq m$. Then from (3.22), $P(k)$ has the form

$$P = \begin{bmatrix} 1 & p^{12} & & \cdots & p^{1n} \\ 0 & 1 & p^{23} & \cdots & p^{2m} \\ \vdots & & & \ddots & \vdots \\ & & & & p^{(m-1)m} \\ 0 & & & \cdots & 1 \end{bmatrix}$$

where p^{ij} are the values in A determined from the elements of the matrix $\bar{G}$. Define

$$\tilde{L} = \begin{bmatrix} \tilde{\ell}_{11} & \cdots & \tilde{\ell}_{1m} \\ \tilde{\ell}_{21} & \cdots & \\ & \ddots & \\ \tilde{\ell}_{m1} & \cdots & \tilde{\ell}_{mm} \end{bmatrix}$$

where

$$\tilde{\ell}_{ii} = [-a_0^{ii} - f_0^{ii} \quad . \; . \; . \quad -a_{n_i-1}^{ii} - f_{n_i-1}^{ii}]$$

$$\underset{i \neq j}{\tilde{\ell}_{ij}} = [-a_0^{ij} \quad . \; . \; . \quad -a_{n_j-1}^{ij}] \quad .$$

Here, f_j^{ii}, $1 \leq i \leq m$, $0 \leq j \leq n_i-1$, are the preassigned coefficients of the system matrix of equation (4.4) and a_ℓ^{ij}, $1 \leq i,j \leq m$, $0 \leq \ell \leq n_j-1$, are the elements of the matrix $\bar{F}$. Then we have

$$(4.6) \quad \bar{F} - \bar{G}P\tilde{L} = \text{block diag} \left\{ \begin{bmatrix} 0 & 1 & 0 & . \; . \; . & 0 \\ \vdots & & & \ddots & \\ & & & & 1 \\ -f_0^{ii} & & . \; . \; . & & -f_{n_i-1}^{ii} \end{bmatrix}, \; n_i x n_i, i=1, \ldots, m \right\}$$

Therefore, if we let $\bar{L}(k) = P(k)\tilde{L}(k)$, the desired feedback gain matrix is

$$(4.7) \qquad \hat{L}(k) = P(k)\tilde{L}(k)T(k)$$

Figure 4.1 shows the block diagram of the feedback system structure for the augmented system. The dynamic controller for the given system (F,G) is the augmentation plus the feedback gain matrix $\hat{L}(k)$. This controller will results in the augmented system having a preassigned closed-loop system dynamic.

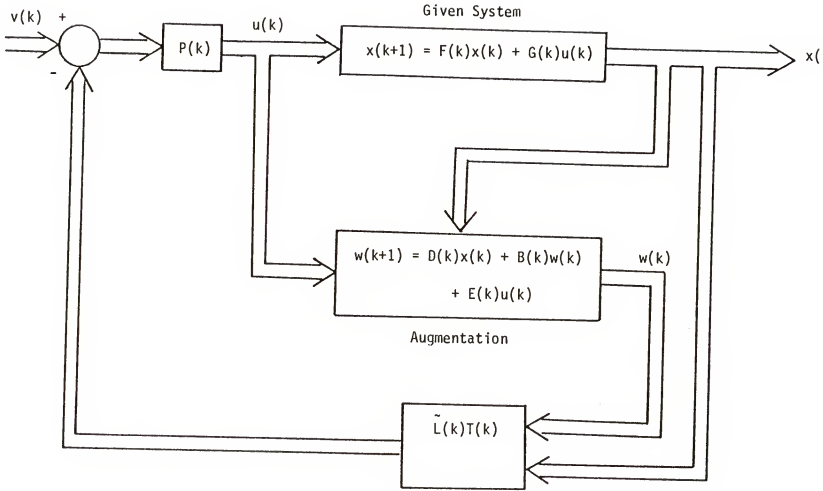


Fig. 4.1 Feedback system structure

We illustrate the above result by the following example.

(4.8) <u>EXAMPLE</u>. Let us consider the augmented system constructed in Example 3.25. The augmented system $(\hat{F},\hat{G})$ is given by

$$\hat{F}(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \hat{G}(k) = \begin{bmatrix} 1 & 1 \\ 0 & od(k) \\ 0 & -1 \end{bmatrix}$$

and the control canonical form $(\bar{F},\bar{G})$ for the augmented system is

$$\bar{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \bar{G} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

where the coordinate transformation $T(k)$ is

$$T(k) = \begin{bmatrix} 1 & -2 & od(k) - ev(k) \\ 0 & 1 & ev(k) \\ 0 & 1 & -od(k) \end{bmatrix}$$

Suppose that we want the closed-loop eigenvalues at 1/2, 0, 1/4. Then

$$\tilde{L} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 1/4 \end{bmatrix}$$

In this case, $P = I_3$ and

$$\hat{L}(k) = \begin{bmatrix} 1/2 & -1 & 1/2\{od(k) - ev(k)\} \\ 0 & 5/4 & ev(k) - 1/4\ od(k) \end{bmatrix}$$

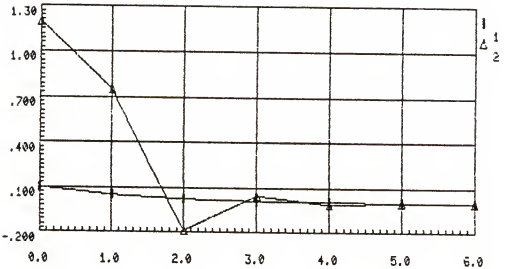
Figure 4.2 shows the simulation results of the above example when $v(k) = 0$ and the initial conditions are $x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $w(0) = 0.2$. []

We now consider the MIMO linear time-varying discrete-time system given by the input/output difference equation

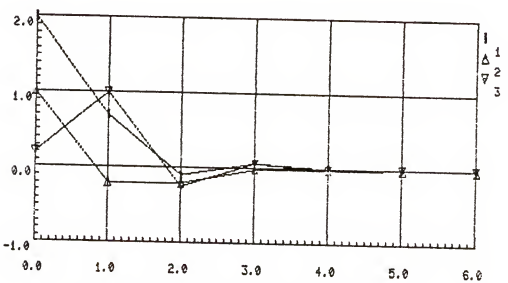
$$(4.9) \qquad y(k+p_1) + \sum_{i=0}^{p_1-1} Q_i(k)y(k+i) = \sum_{i=0}^{p_2-1} R_i(k)u(k+i)$$

where $p_1 > p_2$, Q_i are $p \times p$, and R_i are $p \times m$ time-varying matrices. Equation (4.9) can be rewritten in the following form;

$$(4.10) \qquad y(k+1) = \sum_{i=0}^{p_1-1} Q_i(k-p_1+1)y(k-p_1+1+i) + \sum_{i=0}^{p_2-1} R_i(k-p_1+1)u(k-p_1+1+i)$$



a) Input $u(k) = [u_1(h)\ u_2(k)]'$



b) States $x(k) = [x_1(k)\ x_2(k)]'$ and $w(k)$

Fig. 4.2 Simulation results of Example 4.8 ($v(k) = 0$, $x_1(0) = 2$, $x_2(0) = 2$, $w(0) = 0.2$)

The above equation can be expressed as follows:

$$
\begin{aligned}
y(k+1) &= -Q_{p_1-1}(k-p_1+1)y(k) + r_1(k) \\
r_1(k+1) &= -Q_{p_1-2}(k-p_1+2)y(k) + r_2(k) \\
&\vdots \\
r_{p_1-p_2-1}(k+1) &= -Q_{p_2}(k-p_2)y(k) + r_{p_1-p_2}(k) \\
r_{p_1-p_2}(k+1) &= -Q_{p_2}(k-p_2+1)y(k) + r_{p_2-1}(k-p_2+1)u(k) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + r_{p_1-p_2+1}(k) \\
&\vdots \\
r_{p_1-1}(k+1) &= -Q_0(k)y(k) + r_0(k)u(k)
\end{aligned}
$$

If we define the state vector $x'(k) = [r'_{p_1-1}(k) \; . \; . \; . \; r'(k) \; y'(k)]'$, then we have the state space model

$$
\begin{aligned}
x(k+1) &= F(k)x(k) + G(k)u(k) \\
y(k) &= H(k)x(k)
\end{aligned}
\tag{4.11}
$$

where

$$
F(k) = \begin{bmatrix}
0 & 0 & \cdots & 0 & -Q_0(k) \\
I_p & 0 & \cdots & 0 & -Q_1(k-1) \\
 & & \ddots & & \vdots \\
 & & & I_p & -Q_{p_1-1}(k-p_1+1)
\end{bmatrix}
\tag{4.12a}
$$

$$(4.12b) \qquad G(k) = \begin{bmatrix} R_0(k) \\ R_1(k-1) \\ \vdots \\ R_{p_2-1}(k-p_2+1) \\ 0_{p\times m} \\ \vdots \\ 0_{p\times m} \end{bmatrix}, \quad H = [0_p \quad \ldots \quad 0_p \quad I_p]$$

Let $n = p_1 p$. Then F is $n \times n$, G is $n \times m$, and H is $p \times m$. We note that the above realization is observable in n steps at all times. The observable realization (4.12a,b) is a discrete-time time-varying version of Fuhrmann's realization (see FUHRMANN [1976] and POOLLA [1984]).

If the n-dimensional observable realization (4.12 a,b) is N-step reachable with $N > n$, we can use the augmentation approach to obtain a combined observer-controller compensator which assigns the coefficients of the characteristic polynomial of the closed-loop system arbitrarily.

(4.13) <u>THEOREM</u>. If the m-input n-dimensional time-varying system (F,G,H) defined by (4.12a,b) is N-step reachable for some integer $N > n$, there is an output feedback compensator of dimension q, where $N < q < mN$, that results in the closed-loop system matrix being algebraically equivalent to a constant matrix with q arbitrary assignable eigenvalues.

PROOF. Given the realization (4.12a,b), define

$$M(k) = \begin{bmatrix} -Q_0(k) \\ -Q_1(k) \\ \vdots \\ -Q_{p_1-1}(k-p_1+1) \end{bmatrix}$$

and consider the observer

$$x_0(k+1) = F(k)x_0(k) + M(k)(y(k) - Hx_0(k)) + G(k)u(k) \tag{4.14}$$

If (F,G) is N-step reachable, for some integer q where $N \leq q \leq mN$ there exists a (q-n)-dimensional augmentation

$$w(k+1) = D(k)x_0(k) + B(k)w(k) + E(k)u(k) \tag{4.15}$$

where $D(k)$, $B(k)$, $E(k)$ are determined by the method described in Chapter Three so that $(\hat{F}(k),\hat{G}(k))$ has a fixed basis. From Theorem 3.16, there exists a coordinate transformation $T(k)$ such that $T(k+1)\hat{F}(k)T^{-1}(k)$, $T(k+1)\hat{G}(k)$ are in control canonical form. Then we can choose an m x q feedback matrix $\hat{L}(k)$ so that $T(k+1)(\hat{F}(k) - \hat{G}(k)\hat{L}(k))T^{-1}(k)$ has the preassigned matrix of the form (4.6).

Finally, consider the feedback control law

$$u(k) = -\hat{L}(k) \begin{bmatrix} x_0(k) \\ w(k) \end{bmatrix} \tag{4.16}$$

Let $\hat{L}(k) = [L_1(k)\ L_2(k)]$, where $L_1(k)$ is an m x n matrix and $L_2(k)$ is an m x (q-n) matrix. Combining (4.9), (4.14), (4.15) and (4.16), we have the closed-loop system

$$x_{CL}(k) = F_{CL}x_{CL}(k) + G_{CL}(k)v(k)$$

where

$$x_{CL}(k) = \begin{bmatrix} x(k) \\ w(k) \\ x_0(k) \end{bmatrix} \quad , \quad G_{CL}(k) = \begin{bmatrix} G(k) \\ E(k) \\ G(k) \end{bmatrix}$$

$$F_{CL}(k) = \begin{bmatrix} F & -GL_2 & -GL_1 \\ 0 & B-EL_2 & D-EL_1 \\ -F+MH & 0 & F-MH \end{bmatrix}$$

Define $\bar{x}_{CL}(k) = P^{-1}x_{CL}(k)$, where

$$P^{-1} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_{q-n} & 0 \\ -I_n & 0 & I_n \end{bmatrix}$$

Then we have

$$\bar{x}_{CL}(k+1) = \bar{F}_{CL}(k)\bar{x}_{CL}(k) + \bar{G}_{CL}(k)v(k)$$

where

$$\bar{F}_L = P^{-1}F_{CL}P = \begin{bmatrix} F-GL_1 & -GL_2 & -GL_1 \\ D-EL_1 & B-EL_2 & D-EL_1 \\ 0 & 0 & F-MH \end{bmatrix}$$

$$\bar{G}_{CL} = PG_{CL} = \begin{bmatrix} G(k) \\ E(k) \\ 2G(k) \end{bmatrix}$$

By the discrete-time version of the Lyapunov transformation, $\bar{x}_{CL}$ is algebraically equivalent to x_{CL}. Since F-MH is a fixed matrix and

$$\hat{F} - \hat{G}\hat{L} = \begin{bmatrix} F-GL_1 & -GL_2 \\ D-EL_1 & B-EL_2 \end{bmatrix}$$

can have q arbitrary eigenvalues, the closed-loop system matrix $F_{CL}(k)$ is equivalent to a time-invariant matrix with q assignable eigenvalues.

[]

We now consider the SISO case. The input/output difference equation is given by

$$(4.17) \qquad y(k+n) + \sum_{i=0}^{n-1} a_i(k)y(k+i) = \sum_{i=0}^{n-1} b_i(k)u(k+i) \quad .$$

By a similar derivation as the MIMO case, we can obtain the observable realization (F,g,h) of (4.17) such that

$$(4.18a) \qquad F(k) = \begin{bmatrix} 0 & \cdots & & -a_0(k) \\ 1 & \cdots & 0 & -a_1(k-1) \\ & \ddots & & \vdots \\ & & 1 & -a_{n-1}(k-n+1) \end{bmatrix} \quad .$$

$$(4.18b) \qquad g(k) = \begin{bmatrix} b_0(k) \\ b_1(k-1) \\ \vdots \\ b_{n-1}(k-n+1) \end{bmatrix}, \quad h = [0 \ \cdots \ 0 \ \ 1]$$

Then we have the following theorem which is analogous to Theorem 4.13.

(4.19) <u>THEOREM</u>. If the system (F,g,h) defined by (4.18a,b) is N-step reachable for some integer $N > n$, there is an output feedback compensator of dimension N that results in the closed-loop system matrix being algebraically equivalent to a constant matrix with N arbitrary assignable eigenvalues. []

We have so far considered the construction of output feedback compensators for the time-varying systems given by the input/output difference equations (4.9) and (4.17). Now let us consider any time-varying system that is observable and reachable in finite steps. In the following, we prove the separation property of the observer-controller design procedure when the augmentation approach is used.

Let the system (F,G,H) be given by the state equation

$$
\begin{aligned}
x(k+1) &= F(k)x(k) + G(k)u(k) \\
y(k) &= H(k)x(k)
\end{aligned}
\tag{4.20}
$$

where $x \in A^n$, $y \in A^p$, $u \in A^m$. Suppose that the system (F,G,H) is N_1-step reachable and N_2-step observable, where $N_1, N_2 > n$. Then by the previous results there exists an augmentation for (F,G) such that for some integer q_1 with $N_1 < q_1 < mN_1$, the q_1-dimensional augmented system $(\hat{F}_1, \hat{G})$ is q-step reachable and has a fixed basis, where

$$
\hat{F}_1 = \begin{bmatrix} F & 0 \\ D_1 & B_1 \end{bmatrix}, \qquad \hat{G} = \begin{bmatrix} G \\ E_1 \end{bmatrix}
$$

Let $L(k) = [L_1(k)\ L_2(k)\ ]$ be the feedback gain matrix for the augmented system $(\hat{F},\hat{G})$. Then the closed-loop system matrix will have the form

(4.21) $$\hat{F}_1 - \hat{G}L = \begin{bmatrix} F - GL_1 & -GL_2 \\ D_1 - E_1L_1 & B_1 - E_1L_2 \end{bmatrix}$$

By the duality principle, there also exists an augmentation for (F,H) such that for some integer q_2 with $N_2 < q_2 < pN_2$, the q_2-dimensional augmented system $(\hat{F}_2,\hat{H})$ is q_2-step observable and has a fixed basis, where

$$\hat{F}_2 = \begin{bmatrix} F & D_2 \\ 0 & B_2 \end{bmatrix} \quad , \quad \hat{H} = [H \quad E_2]$$

Consider the observer

(4.22) $$\hat{x}_0(k+1) = \hat{F}_2(k)\hat{x}_0(k) + M(k)(y(k) - \hat{H}(k)\hat{x}_0(k)) + \begin{bmatrix} G(k) \\ 0 \end{bmatrix} u(k)$$

where $\hat{x}_0(k) = \begin{bmatrix} x_0(k) \\ \bar{x}_0(k) \end{bmatrix}$ is the state of the observer, and $M(k) = \begin{bmatrix} M_1(k) \\ M_2(k) \end{bmatrix}$ is the feedback matrix of the observer. Then the closed-loop system matrix of the observer will be

(4.23) $$\hat{F}_2 - M\hat{H} = \begin{bmatrix} F - M_1H & D_2 - M_1E_2 \\ -MH_2 & B_2 - M_2E_2 \end{bmatrix}$$

From the previous results, we have the augmentation and the feedback control law such that

$$
(4.24)\qquad
\begin{aligned}
w(k+1) &= D_1(k)x_0(k) + B_1(k)w(k) + E_1(k)u(k)\\
u(k) &= -L_1(k)x_0(k) - L_2(k)w(k) + v(k)
\end{aligned}
$$

Combining (4.20),(4.22) and (4.24), we have the closed-loop system

$$
(4.25)\qquad x_{CL}(k+1) = F_{CL}(k)x_{CL}(k) + G_{CL}(k)v(k)
$$

where

$$
x_{CL}(k+1) = \begin{bmatrix} x(k+1) \\ w(k+1) \\ x_0(k+1) \\ \bar{x}_0(k+1) \end{bmatrix}, \quad G_{CL}(k) = \begin{bmatrix} G(k) \\ E_1(k) \\ G(k) \\ 0 \end{bmatrix}
$$

$$
F_{CL}(k) = \begin{bmatrix} F & -GL_2 & -GL_1 & 0 \\ 0 & B_1-E_1L_2 & D_1-E_1L_1 & 0 \\ M_1H & -GL_2 & F-M_1H-GL_1 & D_2-M_1E_2 \\ M_2H & 0 & -M_2H & B_2-M_2E_2 \end{bmatrix}
$$

Define $\bar{x}_{CL}(k) = P^{-1}x_{CL}(k)$, where

$$
(4.26)\qquad P^{-1} = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & I_{q_1-n} & 0 & 0 \\ -I_n & 0 & I_n & 0 \\ 0 & 0 & 0 & I_{q_2-n} \end{bmatrix}
$$

Then we have

$$\bar{x}_{CL}(k+1) = \bar{F}_{CL}(k)\bar{x}_{CL}(k) + \bar{G}_{CL}(k)u(k) \tag{4.27}$$

where

$$\bar{F}_{CL} = P^{-1}F_{CL}P = \begin{bmatrix} F - GL_1 & -GL_1 & -GL_1 & 0 \\ D_1 - E_1L_1 & B_1-E_1L_2 & D_1 & 0 \\ 0 & 0 & F-M_1H & D_2-M_1E_2 \\ 0 & 0 & -M_2H & B_2-M_2E_2 \end{bmatrix}$$

$$\bar{G} = PG_{CL} = \begin{bmatrix} G \\ E_1 \\ 2G \\ 0 \end{bmatrix}$$

By the discrete-time version of the Lyapunov transformation, $\bar{x}_{CL}$ is algebraically equivalent to x_{CL} and from (4.21),(4.23) and (4.27) we see that the controller and the observer can be designed separately. This is the separation property of the observer-controller design procedure. Figure 4.3 shows the structure of the combined observer-controller feedback system.

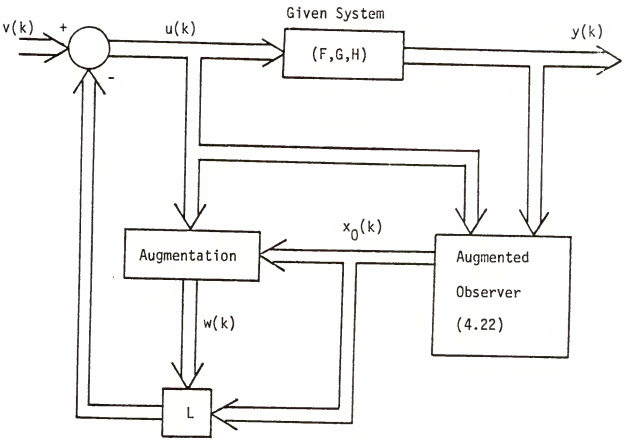


Fig. 4.3 Combined observer-controller structure

CHAPTER FIVE
CYCLIZABILITY

In this chapter, we shall use the augmentation approach to derive a different constructive method of the design of feedback control systems for N-step reachable time-varying systems from Chapter Four. Since the augmented system constructed in Chapter Three has a fixed basis, it was possible to obtain a generalized control canonical form which led to the design of a feedback control system which results in the closed-loop system matrix being algebraically equivalent to a constant matrix with arbitrary assignable eigenvalues. However, it may be quite laborious to transform the system to the control canonical form. It is therefore of interest to obtain a more direct method.

If a time-invariant multivariable system is reachable, then there exists a feedback gain matrix so that the closed-loop system is reachable through one of the inputs. Therefore, it is possible to construct feedback control systems by using single-input methods without having to consider the multivariable control canonical form. Moreover, we can achieve the coefficient assignability by this method. This notion, which is called cyclizability, was first introduced by HEYMANN [1968]. It is well known that a time-invariant system is reachable if and only if it is cyclizable (see HEYMANN [1968] and HAUTUS [1977]).

However, in the time-varying case, the notion of cyclizability has not been thoroughly investigated, and there are still several open questions. For example, if a time-varying system is cyclizable, then it is always reachable. But, it is not known whether or not reachability implies cyclizability.

To study the time-varying case, we begin with the time-varying version of the definition of cyclizability.

An m-input n-dimensional time-varying system (F,G) is said to be <u>N-cyclizable with N > n</u> if there exist a matrix $L_0 \varepsilon A^{m \times n}$ and a vector $u_0 \varepsilon A^m$ such that $(F-GL_0, Gu_0)$ is N-cyclic. If $(F-GL_0, Gu_0)$ is n-cyclic, then the system is said to be <u>n-cyclizable</u>. We note that there is no notion of N-cyclizability in the time-invariant case. In the following lemma, KAMEN and HAFEZ [1979] have obtained a necessary and sufficient condition for n-cyclizability in the time-varying case.

(5.1) <u>LEMMA</u>. There exist a matrix $L_0 \varepsilon A^{m \times n}$ and a vector $u_0 \varepsilon A^m$ such that $(F-GL_0, Gu_0)$ is n-cyclic (or n-step reachable) if and only if there exist $u_0, u_1, \ldots, u_{n-1} \varepsilon A^m$ such that

$$\text{rank } [\xi_0 \ \xi_1 \ \ldots \ \xi_{n-1}] = n \quad \text{at all times}$$

where

$$\xi_0 = Gu_0 ,$$

$$\xi_i = F(\sigma\xi_{i-1}) + Gu_i \quad , \quad i=1,2, \ldots , n-1. \qquad []$$

However, in the time-varying case, n-step reachability of the system (F,G) does not always imply that (F,G) is n-cyclizable as we see in the following lemma.

(5.2) LEMMA. Suppose that the system (F,G) is n-step reachable. If $G(k) = 0$ for at least one value of k, then (F,G) is not n-cyclizable.

PROOF. For the system (F,G) to be n-cyclizable, there must exist a matrix $L_0 \in A^{m \times n}$ and a vector $u_0 \in A^m$ such that $(F-GL_0, Gu_0)$ is n-cyclic. If $G(k_1) = 0$ for some integer k_1, then $G(k_1)u_0 = 0$ for any vector $u_0 \in A^m$. Therefore, for the vectors ξ_i defined in Lemma 5.1, rank $[\xi_0\ \xi_1\ \ldots\ \xi_{n-1}] \neq n$ at time k_1, and thus the system (F,G) is not n-cyclizable. []

However, even if (F,G) is not n-cyclizable, it may be N-cyclizable for some finite integer $N > n$. In other words, there may exist an L_0 and an u_0 such that $(F-GL_0, Gu_0)$ is N-cyclic when the system is n-step reachable.

(5.3) EXAMPLE. Let $F(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $G(k) = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$.

Then $[\ G\ \ S_F G\ ] = \begin{bmatrix} k & 0 & k-1 & 0 \\ 0 & k & 0 & k-1 \end{bmatrix}$ which has rank 2 for integers k.

Thus the system is 2-step reachable. But, since $G(k) = 0$ at $k = 0$, by Lemma 5.2, (F,G) is not 2-cyclizable. However, if we choose

$$L_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

then $F-GL_0 = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}$ and $Gu_0 = \begin{bmatrix} k \\ 0 \end{bmatrix}$.

The system $(F-GL_0, Gu_0)$ is 3-step reachable. Therefore, the system (F,G) is 3-cyclizable. []

Now suppose that the system (F,G) is N-step reachable and M-cyclizable, where $M > N > n$. That is, let us suppose that there exist an L_0 and an u_0 such that $(F-GL_0, Gu_0)$ is M-step reachable. Then from the augmentation results of the single-input case in Chapter Three, there exist an $(M-n) \times n$ matrix D, an $(M-n) \times (M-n)$ matrix B and an $(M-n) \times 1$ vector e such that the system $(\hat{F},\hat{g})$ is M-step reachable where

$$\hat{F} = \begin{bmatrix} F-GL_0 & 0 \\ D & B \end{bmatrix}, \qquad \hat{g} = \begin{bmatrix} Gu_0 \\ e \end{bmatrix}.$$

In this case, the closed-loop feedback system for (F,G) has the structure in Fig. 5.1, where $[\ell_1 \quad \ell_2]$ is the feedback gain matrix for the single-input augmented system $(\hat{F},\hat{g})$ and $u_a(k) = -\ell_1(k)x(k) - \ell_2(k)w(k)$.

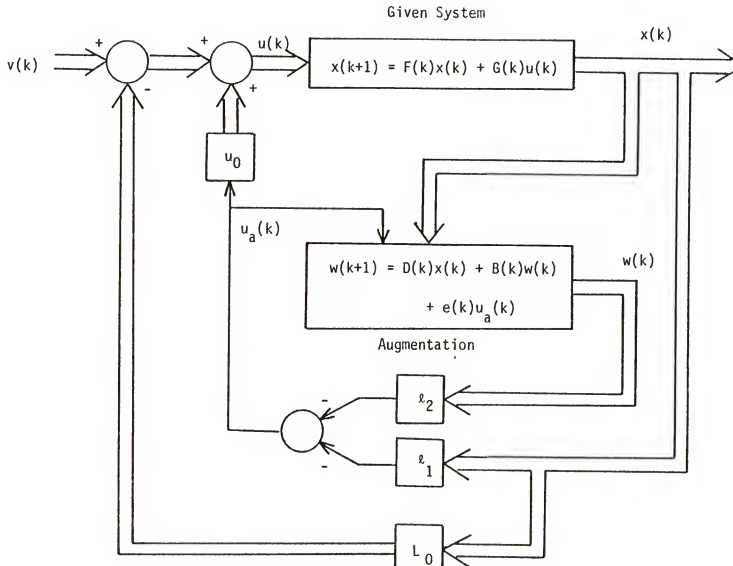


Fig. 5.1 Augmentation for M-cyclizable systems

We have so far considered cyclizability of N-step reachable time-varying systems. But, we can not use this result to construct a feedback system for arbitrary N-step reachable systems since it is unknown whether or not N-step reachability always implies M-cyclizability with $M > N > n$.

We now consider cyclizability of the augmented system constructed in Chapter Three. In the following, we shall show that the augmented system constructed in Chapter Three is always cyclizable.

(5.4) THEOREM. Suppose that an m-input n-dimensional system (F,G) is N-step reachable with $N > n$ and that for some integer q where $N < q < mN$, $(\hat{F},\hat{G})$ is the q-dimensional augmented system constructed in the proof of Theorem 3.9. Then $(\hat{F},\hat{G})$ is q-cyclizable. That is, there exist a matrix $L_0 \in A^{m \times q}$ and a vector $u_0 \in A^m$ such that $(\hat{F}-\hat{G}L_0, \hat{G}u_0)$ is q-step reachable.

PROOF. For the simplicity of the proof, let us choose $\hat{g}_1$ as a generator. Then $u_0 = [1 \;\; 0 \; . \; . \; . \; 0]'$ and $\hat{G}u_0 = g_1$. From the generalized controllability indices determined in the proof of Theorem 3.10, let

$$\alpha_{max} = \max_{1 < i < m} \{n_i\} \quad .$$

In the following, we let $n_0 = 0$.

We now define integers t and j_t as follows:

<u>Initialization</u>: $t = 0$, $i = 1$, $\alpha = 0$.

<u>Step 1</u> : If $n_\alpha > i$, then $j_t = (\sum_{\ell=1}^{\alpha} n_{\ell-1}) + i$, increase t by 1, and go to Step 2.

If $n_\alpha < i$, then go to Step 2.

<u>Step 2</u> : If $\alpha < m$, then increase α by 1 and go to Step 1.

If $\alpha = m$ and if $i < \alpha_{max}$, then increase i by 1 and go to Step 1.

If $\alpha = m$ and if $i = \alpha_{max}$, then we end this algorithm.

From the integers $1 \leq t \leq q$, $1 \leq j_t \leq q$, we let P be a q x q matrix whose t^{th}-column vector is a q x 1 unit vector having 1 at $j_t{}^{th}$ position. The matrix P is called a permutation matrix.

From the coefficients $r_j^{\ell i} \in A$ of (3.12), we define the elements $a_{n_j^\ell} \in A$ as follows:

$$a_{n_j}^{\ell} = \begin{cases} \sigma^{-(n_j+1)} (r_{j+1}^{\ell(j+1)}) & , \quad 1 \leq \ell \leq j \\ 1 & , \quad \ell = j+1 \\ 0 & , \quad j+2 \leq \ell \leq m \end{cases}$$

for $\ell = 1,2, \ldots, m$; $j = 1,2, \ldots, m-2$

$$a_{n_{m-1}}^{\ell} = \begin{cases} 0 & , \ell = 1,2, \ldots, m-1 \\ 1 & , \ell = m. \end{cases}$$

Now define the vectors $\tilde{u}_{ij} \in A^m$ for $i = 1, 2, \ldots, m-1$, $j=1,2, \ldots, n_i$ as follows :

$$\tilde{u}_{ij} = \begin{cases} \begin{bmatrix} r_0^{1i} + a^1_{n_i} \\ \vdots \\ r_0^{mj} + a^m_{n_i} \end{bmatrix} & , \text{ when } j = n_i \\ \\ \begin{bmatrix} r_{n_i-j}^{1i} \\ \vdots \\ r_{n_i-i}^{mi} \end{bmatrix} & , \text{ otherwise} \end{cases}$$

For $i = 1, 2, \ldots, m-1$, $j = 1, 2, \ldots, n$, let $\beta = (\sum_{\ell=1}^{i} n_{\ell-1}) + j$. Define $u_\beta = \tilde{u}_{ij}$, for $1 \leq i \leq m-1$, $1 \leq j \leq n_i$ and $u_\beta = 0$ for $n_1 + n_2 + \ldots n_{m-1} + 1 \leq \beta \leq q-1$.

Define $\xi_0, \xi_1, \ldots, \xi_{q-1}$ as

$$\xi_0 = \hat{G}u_0$$
$$\xi_i = \hat{F}\sigma\xi_{i-1} + \hat{G}u_i, \; i = 1, 2, \ldots, q-1.$$

Then it can be shown that

$$[\xi_0 \quad \xi_1 \quad \ldots \xi_{q-1}]P = (\tilde{C}_q P)U_q$$

where

$$\tilde{C}_q = [\hat{g}_1 \quad . \; . \; . \; S_{\hat{F}}^{n_1-1}\hat{g}_1 \quad \hat{g}_2 \quad . \; . \; . \; \hat{g}_m \quad . \; . \; . \; S_{\hat{F}}^{n_m-1}\hat{g}_m]$$

and $U_q \in A^{q \times q}$ is the upper triangular matrix whose diagonal elements are 1's.

Since P, $\tilde{C}_q$, U_q have rank q at all times,

$$\text{rank } [\xi_0 \quad \xi_1 \quad . \; . \; . \; \xi_{q-1}] = q \quad \text{at all times.}$$

Let $L_0 = -[u_1 \; u_2 \; . \; . \; . \; u_{q-1} \; 0]\sigma[\xi_0 \quad \xi_1 \; . \; . \; . \; \xi_{q-1}]^{-1}$. Then the reachability matrix of the system $(\hat{F}-\hat{G}L_0, \hat{G}u_0)$ is $[\xi_0 \quad \xi_1 \; . \; . \; . \; \xi_{q-1}]$. Therefore, $(\hat{F}-\hat{G}L_0, \hat{G}u_0)$ is q-step reachable and thus the theorem is proved. []

Again let (F,G) be an N-step reachable system with reachability matrix $C_N(k)$, and consider the array of vectors shown below :

$$\begin{array}{cccc}
g_1 & g_2 & . \; . \; . & g_m \\
S_F g_1 & S_F g_2 & . \; . \; . & S_F g_m \\
\vdots & \vdots & \ddots & \\
S_F^N g_1 & S_F^N g_1 & . \; . \; . & S_F^N g_m
\end{array}$$

By searching the columns of this array <u>from top to bottom</u>, we select those vectors that are not equal to an A-linear combination of

previously selected vectors. Vectors that are A-linear combinations of previously selected vectors are discarded. In this selection, we assume that there exist an integer t (where $1 \leq t \leq m$) and integers $n_1, n_2, \ldots, n_t$ (where $1 \leq n \leq N$ for $i = 1, 2, \ldots, t$) such that

$$(5.5) \qquad S_F^{n_i} g_i = - \sum_{j=1}^{i} \sum_{\ell=0}^{n_j-1} r_{j\ell}^{i} (S_F^{\ell} g_j) \quad , \quad i = 1, 2, \ldots, t$$

where $r_{j\ell}^{i}$ are elements in A, and that the vectors $g_1, S_F g_1, \ldots, S_F^{n_1-1} g_1, g_2, \ldots, g_t, \ldots, S_F^{n_t-1} g_t$ generate A^n. Here, $g_1, g_2, \ldots, g_t$ are all included in our selection. If this is not the case, then we can rearrange the above array by reordering those columns. Therefore, we can wok with the reordered array. Letting $q = n_1 + n_2 + \ldots + n_t$, we have $N \leq q \leq mN$.

If the system (F,G) is time-invariant, there always exist integers $t, n_1, n_2, \ldots, n_t$ such that (5.5) holds (see KAILATH [1980]). However, in the time-varying case, there may not exist $S_F^{j} g_i$ for all $1 \leq i \leq m$ and for all $1 \leq j \leq N$ such that

$$S_F^{j} g_i = - \sum_{r=1}^{i=1} \sum_{\ell=0}^{j-1} \alpha_{\ell}^{r} (S_F^{\ell} g_i) \quad , \quad \alpha_{\ell}^{r} \in A \quad .$$

Therefore, we are considering a very restricted case of Theorem 5.4. However, this restriction gives a very simple constructive proof of cyclizability.

We now consider the augmentation for the system (F,G) which is assumed as above.

Choose q vectors $e_{ij}(k)$, $1 \leq j \leq t$, $0 \leq i \leq n_j-1$, where e_{ij} are $(q-n) \times 1$ vectors, such that

$$\text{rank} \begin{bmatrix} g_1 & \cdots & S_F^{n_1-1} g_1 & g_2 & \cdots & g_t & \cdots & S_F^{n_t-1} g_t \\ e_{01} & \cdots & e_{(n_1-1)1} & e_{02} & \cdots & e_{0t} & \cdots & e_{(n_t-1)t} \end{bmatrix} = q$$

at all times.

This is always possible from Proposition 3.4. Let $\tilde{C}_q(k)$ denote the $q \times q$ matrix

$$(5.6) \quad \tilde{C}_q(k) = \begin{bmatrix} g_1 & \cdots & S_F^{n_1-1} g_1 & g_2 & \cdots & g_t & \cdots & S_F^{n_t-1} g_t \\ e_{01} & \cdots & e_{(n_1-1)1} & e_{02} & \cdots & e_{0t} & \cdots & e_{(n_t-1)t} \end{bmatrix}$$

From (5.5) and (5.6), calculate $e_{n_i i}$, $i = 1,2, \ldots ,t$, such that

$$(5.7) \qquad e_{n_i i} = - \sum_{j=1}^{i} \sum_{\ell=0}^{n_j-1} r_{j\ell}^{i} e_{\ell j} \ .$$

Define $E(k) = [e_{01}(k)\ e_{02}(k) \ldots e_{0t}(k)\ * \ \ldots *]$ and

$$[D(k)\ B(k)] = X(k)(\sigma\tilde{C}^{-1})(k)$$

where $*$'s are arbitrary vectors and $X(k) = [e_{11} \ldots e_{n_1 1}\ e_{12} \ldots e_{1t} \ldots e_{n_t t}]$.

Let $\hat{F} = \begin{bmatrix} F & 0 \\ D & B \end{bmatrix}$ and $\hat{G} = \begin{bmatrix} G \\ E \end{bmatrix}$. Then by the same argument in the proof of theorem 3.10, we see that the q-dimensional augmented system $(\hat{F},\hat{G})$ is q-step reachable at all times. In the following theorem, we shall show that the augmented system constructed above is q-cyclizable.

(5.8) <u>THEOREM</u>. Let (F,G) be an N-step reachable system having the property (5.5). Then for some integer q where $N \leq q \leq mN$ the augmented system $(\hat{F}, \hat{G})$ constructed above is q-cyclizable, i.e. there exist a matrix $L_0 \in A^{m \times q}$ and a vector $u_0 \in A^m$ such that $(\hat{F} - \hat{G}L_0, \hat{G}u_0)$ is q-step reachable.

<u>PROOF</u>. Without loss of generality, we choose $\hat{g}_1$ as a generator. Then $u_0 = [1 \;\; 0 \;\; . \; . \; . \; 0]'$. Define

$$\xi_0 = \hat{G}u_0$$
$$\xi_i = \hat{F}(\sigma\xi_{i-1}) + \hat{G}u_i, \; i = 1, 2, \ldots, q-1$$

where

$$u_i = \begin{cases} [0 \; . \; . \; . \; \underset{\substack{\uparrow \\ (i-n+1)^{th}}}{1} \; . \; . \; . \; 0]', & \text{when } i = n_1, n_2, \ldots, n_t \\ 0 & \text{, otherwise.} \end{cases}$$

Then we have

$$
\begin{aligned}
\xi_0 &= \hat{G}u_0 = \hat{g}_1 \\
&\vdots \\
\xi_{n_1-1} &= S_{\hat{F}}^{n_1-1}\hat{g}_1 \\
\xi_{n_1} &= \hat{F}(\sigma\xi_{n_1-1}) + \hat{g}_2 = \hat{g}_2 + S_{\hat{F}}^{n_1}\hat{g}_1 \\
&\vdots \\
\xi_{q-n_t+1} &= \hat{g}_t + \cdots \\
\xi_{q-1} &= S_{\hat{F}}^{n_t-1}\hat{g}_t + \cdots
\end{aligned}
$$

If we rewrite the above vectors in a matrix format, then we obtain

$$[\xi_0 \;\; \xi_1 \; \cdots \; \xi_{q-1}] = \tilde{C}_q \, U_q$$

where $\tilde{C}_q$ is given by (5.6) and U_q is an upper triangular matrix whose diagonal elements are 1's. Since $\tilde{C}_q$, U_q are invertible, $[\xi_0 \;\; \xi_1 \; \cdots \; \xi_{q-1}]$ is also invertible. Define

$$L_0 = -[u_1 \;\; u_2 \; \cdots \; u_{q-1} \;\; 0\,]\sigma[\xi_0 \;\; \xi_1 \; \cdots \; \xi_{q-1}]^{-1} \quad .$$

Then the reachability matrix of the system $(\hat{F}-\hat{G}L_0, \hat{G}u_0)$ is $[\xi_0 \;\; \xi_1 \; \cdots \; \xi_{q-1}]$ and thus the theorem is proved. []

We now construct a feedback gain matrix for the augmented system $(\hat{F},\hat{G})$ by the method discussed in this chapter. Since the single-input system $(\hat{F}-\hat{G}L_0,\hat{G}u_0)$ constructed in Theorem 5.4 and Theorem 5.8 is q-step reachable, from (2.8) there exists a coordinate transformation $T(k)$ such that

$$\begin{aligned} \bar{F}(k) &= T(k+1)(\hat{F}(k) - G(k)L(k))T^{-1}(k) \\ \bar{g}(k) &= T(k+1)\hat{G}u_0 \end{aligned}$$

are in the control canonical form. Let us define $a_0, a_1, \ldots, a_{q-1} \in A$ as

$$(5.9)\quad [a_0 \;\; a_1 \;.\;.\; a_{q-1}]' = -[\xi_0 \;\; \xi_1 \;.\;.\;.\; \xi_{q-1}]^{-1} \left(S_{(\hat{F}-\hat{G}L_0)}^{\;q}(\hat{G}u_0)\right) \quad .$$

We pick any monic polynomial $\chi(z)$ of degree q as a characteristic polynomial of the closed-loop system for $(\hat{F}-\hat{G}L_0,\hat{G}u_0)$, where $\chi(z)$ has the form

$$(5.10)\qquad \chi(z) = z^q + \sum_{i=0}^{q-1} z^i b_i \quad , \quad b_i \in \mathbf{R} \quad .$$

Then $\ell(k) = -[b_0 - a_0 \;.\;.\;.\; b_{q-1} - a_{q-1}]T(k)$ is the feedback gain matrix for the single-input system $(\hat{F}-\hat{G}L_0,\hat{G}u_0)$ which makes the system have a characteristic polynomial $\chi(z)$. Therefore, if we let

(5.11) $$L(k) = L_0(k) - u_0\ell(k) \quad ,$$

then $L(k)$ is the feedback gain matrix for the augmented system $(\hat{F},\hat{G})$, and the closed-loop system will have the characteristic polynomial $\chi(z)$.

We illustrate the above results by the following example.

(5.12) <u>EXAMPLE</u>. Consider the system (F,G) such that

$$F(k) = \begin{bmatrix} 1 & 0 \\ -od(k) & 1 \end{bmatrix} \quad \text{and} \quad G(k) = \begin{bmatrix} 1 & 0 \\ ev(k) & od(k) \end{bmatrix}$$

Then the reachability matrix is

$$C_2(k) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ ev(k) & od(k) & 0 & ev(k) \end{bmatrix}$$

and this system is 2-step reachable. We select g_1, g_2, $S_F g_1$ from $C_2(k)$ as the vectors that are not equal to an A-linear combination of previously selected vectors. Then $n_1 = 2$, $n_2 = 1$ and $q = 3$. Now determine e_{01}, e_{02}, e_{11} such that

$$\tilde{C}_3(k) = \begin{bmatrix} g_1 & g_2 & S_F g_1 \\ e_{01} & e_{02} & e_{11} \end{bmatrix}$$

has rank 3 at all times.

Letting $e_{01} = -od(k)$, $e_{02} = ev(k)$, $e_{11} = 0$, we have rank $\tilde{C}_3(k) = 3$ at

all times. From $S_F{}^2 g_1 = -g_2 + S_F g_1$ and $S_F g_2 = g_1 - S_F g_1$, calculate e_{21} and e_{12}. Then we have

$$e_{21} = -od(k)$$

$$e_{12} = -ev(k)$$

Define $E(k) = [e_{01} \;\; e_{02}] = [-od(k) \;\; ev(k)]$ and

$$\begin{aligned}
[D(k) \;\; B(k)] &= [e_{11} \;\; e_{21} \;\; e_{12}](\sigma\tilde{C}_3)^{-1}(k) \\
&= [0 \;\; -od(k) \;\; -ev(k)] \begin{bmatrix} 0 & od(k) & -ev(k) \\ 0 & ev(k) & od(k) \\ 1 & -od(k) & ev(k) \end{bmatrix} \\
&= [-ev(k) \;\; 0 \;\; -1\,].
\end{aligned}$$

Then the augmented system $(\hat{F},\hat{G})$ has the form

$$\hat{F} = \begin{bmatrix} 1 & 0 & 0 \\ -od(k) & 1 & 0 \\ -ev(k) & 0 & -1 \end{bmatrix} \quad \text{and} \quad \hat{G} = \begin{bmatrix} 1 & 0 \\ ev(k) & od(k) \\ -od(k) & ev(k) \end{bmatrix}$$

Now, let us construct a feedback control system for the augmented system $(\hat{F},\hat{G})$. Let

$$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Then $\xi_0 = \hat{G}u_0 = \hat{g}_1 = \begin{bmatrix} 1 \\ ev(k) \\ -od(k) \end{bmatrix}$

Take $u_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. Then we have

$$\xi_1 = \hat{F}(\sigma\hat{g}_1) - \hat{g}_1 = \begin{bmatrix} 0 \\ -ev(k) \\ od(k) \end{bmatrix}$$

$$\xi_2 = \hat{F}(\sigma\xi_1) + \hat{g}_2 = \begin{bmatrix} 0 \\ od(k) \\ ev(k) \end{bmatrix}$$

Since $[\xi_0 \;\; \xi_1 \;\; \xi_2]$ is invertible, we can define

$$\begin{aligned} L_0 &= - [u_1 \; u_2 \; 0]\sigma[\xi_0 \;\; \xi_1 \;\; \xi_2]^{-1} \\ &= - \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -od(k) & ev(k) \\ 0 & ev(k) & od(k) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2od(k) & -2ev(k) \end{bmatrix} \end{aligned}$$

From (5.9), we have $a_0 = 0$, $a_1 = 1$, $a_2 = 0$. Therefore, the coordinate transformation matrix T(k) will be

$$T(k) = \begin{bmatrix} 1 & -od(k) & ev(k) \\ 1 & -ev(k) & ev(k) \\ 0 & ev(k) & od(k) \end{bmatrix}$$

Pick $\chi(z) = z^3 - z0.25$. Then from (5.11) we have

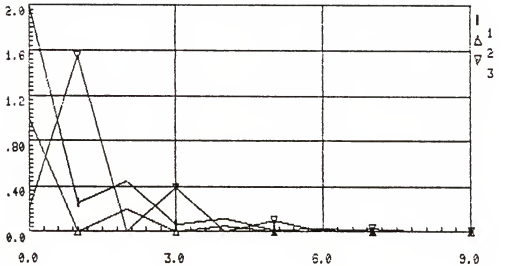
$$\begin{aligned} \hat{L}(k) &= L_0 + u_0[b_0-a_0 \;\; b_1-a_1 \;\; b_2-a_2]T(k) \\ &= \begin{bmatrix} 0.75 & 0.25od(k) & -0.25ev(k) \\ -1 & od(k) & -ev(k) \end{bmatrix} \end{aligned}$$

If we construct the feedback gain matrix from the generalized control canonical form described in Chapter Three and Chapter Four, then we have

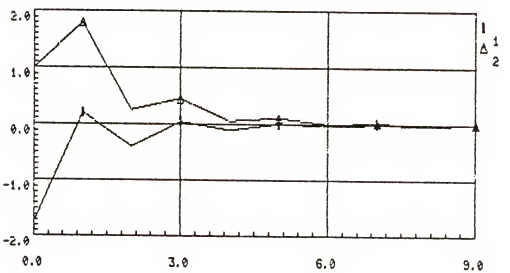
$$\hat{L}(k) = \begin{bmatrix} 1 + 1.75(od(k) - ev(k)) & -1.75od(k) & -1.75ev(k) \\ od(k) - ev(k) & -od(k) & -ev(k) \end{bmatrix}$$

The simulation results are given in Fig. 5.2 and Fig. 5.3 when $v(k) = 0$ and the initial conditions are $x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $w(0) = 0.2$.

It is interesting to compare the responses of the closed-loop systems constructed by those two methods. []

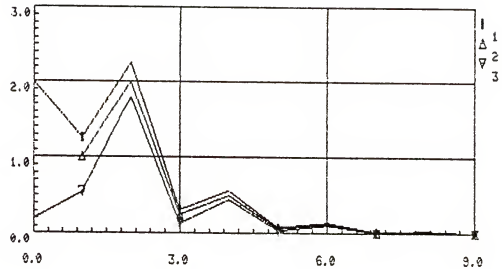


a) States $x(k) = [x_1(k)\ x_2(k)]'$ and $w(k)$

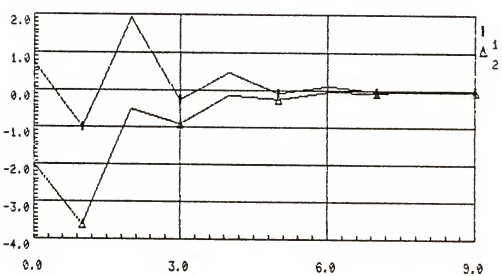


b) Input $u(k) = [u_1(k)\ u_2(k)]'$

Fig. 5.2 Simulation results of Example 5.12 (Canonical form method: $v(k) = 0$, $x_1(0) = 2$, $x_2(0) = 1$, $w(0) = 0.2$)



a) States x(k) and w(k)



b) Input u(k)

Figure 5.2 Simulation results of Example 5.12 (Direct method: $v(k) = 0$, $x_1(0) = 2$, $x_2(0) = 1$, $w(0) = 0.2$)

CHAPTER SIX
TIME-VARYING RESULTANT MATRICES

The second part of this dissertation is concerned with developing the theory of the resultant for linear time-varying systems. The resultant is a classical tool used in examining the relative primeness of a prescribed pair of polynomials. In the time-invariant case, the resultant has a wide range of applications in system theory especially when the system is given by a polynomial matrix-fraction description. For example, by obtaining greatest common divisors of polynomials and polynomial matrices via the resultant, we can determine uncontrollable and unobservable modes of given systems.

However, little attention has been paid to the development of the resultant in the time-varying case since a polynomial matrix-fraction description for time-varying systems was not available until KAMEN et al. [1985] and POOLLA [1984] successfully developed a time-varying version of the polynomial matrix-fraction approach in the framework of skew rings of polynomials, formal power series, and formal Laurent series. Moreover, their results closely resemble the time-invariant case. Using their results, we define the resultant for time-varying systems and exploit several applications to system theory in the second part of this dissertation.

In this chapter, we define a time-varying version of Sylvester's resultant matrix for polynomials and polynomial matrices. It will be shown that reachability of the observable realization (4.12a,b) and (4.18a,b) can be checked by the resultant matrix. We begin with the definition of the resultant matrix for two polynomials with time-varying coefficients.

Let a pair of polynomials with time-varying coefficients, a(z) and b(z), be given by

$$
\begin{aligned}
a(z) &= z^n + z^{n-1} a_{n-1}(k) + \ldots + za_1(k) + a_0(k) \\
b(z) &= z^{n-1} b_{n-1}(k) + \ldots + zb_1(k) + b_0(k) \quad .
\end{aligned}
\tag{6.1}
$$

Here, we should note that a(z) and b(z) are elements of right skew polynomial ring A[z] defined in Chapter Two. The indeterminate z has the noncommutative property $\alpha z = z(\sigma\alpha)$, $\alpha \in A$, where σ is the right-shift operator such that $(\sigma\alpha)(k) = \alpha(k-1)$. Then we have the following time-varying version of the resultant matrix.

(6.2) <u>DEFINITION</u>. The time-varying resultant matrix $\Gamma_n(a,b)$ of two polynomials a(z) and b(z) given by (6.1) is defined as

$$\Gamma_n(a,b) = \begin{bmatrix} a_0 & 0 & \cdots & b_0 & 0 & \cdots \\ a_1 & \sigma a_0 & & b_1 & \sigma b_0 & \\ \vdots & & \ddots \; \sigma^{n-2}a_0 & \vdots & & \ddots \\ a_{n-1} & & & b_{n-1} & & \sigma^{n-1}b_0 \\ 1 & \sigma a_{n-1} & \vdots & 0 & \sigma b_{n-1} & \\ & 1 & \ddots & \vdots & & \vdots \\ \vdots & \ddots & \sigma^{n-2}a_{n-1} & 0 & & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & \sigma^{n-1}b_{n-1} \end{bmatrix}$$

where $\Gamma_n(a,b) \in A^{(2n-1) \times (2n-1)}$. []

The above definition closely resembles Sylvester's resultant matrix in the time-invariant case (see BARNETT [1983]). In the following, we relate the definition of the time-varying resultant matrix to reachability of a given system.

Let us consider the input/output difference equation given by

$$(6.3) \qquad y(k+n) + \sum_{i=0}^{n-1} a_i(k)y(k+i) = \sum_{i=0}^{n-1} b_i(k)u(k+i) \quad .$$

The observable realization (F,g,h) of equation (6.3) is given by (4.18a,b). Define polynomials related to the input/output difference equation (6.3) as follows:

$$a(z) = z^n + z^{n-1}a_{n-1}(k-n+1) + \ldots + za_1(k-1) + a_0(k)$$

(6.4)

$$b(z) = z^{n-1}b_{n-1}(k-n+1) + \ldots + zb_1(k-1) + b_0(k) \ .$$

Then reachability of the observable realization (F,g,h) of (6.3) can be examined by the resultant matrix of the two polynomials a(z), b(z) given by (6.4).

(6.5) <u>THEOREM</u>. The n-dimensional observable realization (F,g,h) given by (4.18a,b) is n-step reachable at all times if and only if

$$\det \Gamma_n(a,b) \neq 0 \quad \text{at all times}$$

where a(z), b(z) are given by (6.4) and det denotes the determinant of a matrix.

<u>PROOF</u>. First, partition the resultant matrix $\Gamma_n(a,b)$ as follows:

$$\Gamma_n(a,b) = \begin{array}{cc} \begin{array}{cc} n-1 & n \end{array} & \\ \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_3 & \Gamma_4 \end{bmatrix} & \begin{array}{c} n \\ n-1 \end{array} \end{array}$$

Postmultiply $\Delta = \begin{bmatrix} \Delta_1 & I_{n-1} \\ I_n & 0 \end{bmatrix}$ on both sides of the above equation,

where Δ_1 is an (n-1) x n matrix satisfying the following:

$$\Gamma_3\Delta_1 + \Gamma_4 = 0$$

Then we have

$$\Gamma_n(a,b)\Delta = \begin{bmatrix} \Gamma_1\Delta_1 + \Gamma_2 & \Gamma_1 \\ 0 & \Gamma_3 \end{bmatrix} . \tag{6.6}$$

Since Γ_3 is an invertible (n-1) x (n-1) matrix, there always exists a matrix Δ_1 such that $\Gamma_3\Delta_1 + \Gamma_4 = 0$. It remains to show that

$$\Gamma_1\Delta_1 + \Gamma_2 = C_n$$

where $C_n = [g\ S_F g\ .\ .\ .\ S_F^{n-1}g]$. This is done by comparing columns on both sides of the above equation with $\Delta_1 = -\Gamma_3^{-1}\Gamma_4$. From (6.6), it can be shown that

$$\begin{aligned} \det \Gamma_n(a,b) &= \det \Delta \det C_n \\ &= (-1)^{n(n-1)/2} \det C_n \quad . \end{aligned}$$

Hence, the system (F,g,h) is n-step reachable at all times if and only if $\det \Gamma_n(a,b) \neq 0$ at all times. []

In the time-varying case, an n-dimensional system can be N-step reachable with N > n. Therefore, we need to generalize Theorem 6.5 to the N-step reachable case. To do this, we modify Definition 6.2 as follows:

(6.7) <u>DEFINITION</u>. For an integer $N \geq n$, the time-varying resultant matrix $\Gamma_N(a,b)$ of two polynomials $a(z)$ and $b(z)$ given by (6.1) is defined as

$$\Gamma_N(a,b) = \begin{bmatrix} a_0 & 0 & \cdots & b_0 & 0 & \cdots & \\ a_1 & \sigma a_0 & & b_1 & \sigma b_0 & & \\ \vdots & & \ddots & \vdots & & & \\ & & \sigma^{N-2}a_0 & & & \ddots & \\ a_{n-1} & & & b_{n-1} & & & \sigma^{N-1}b_0 \\ 1 & \sigma a_{n-1} & \vdots & 0 & \sigma b_{n-1} & & \\ \vdots & 1 & \ddots & \vdots & & & \\ & & \sigma^{N-2}a_{n-1} & & & \ddots & \\ 0 & \cdots & 1 & 0 & \cdots & & \sigma^{N-1}b_{n-1} \end{bmatrix}$$

where $\Gamma_N(a,b) \in A^{(N+n-1) \times (2N-1)}$. []

When $N=n$, the above definition is exactly same as Definition 6.2. But, if $N > n$, then the resultant matrix is no longer a square matrix.

(6.8) <u>THEOREM</u>. The n-dimensional observable realization (F,g,h) given by (4.18 a,b) is N-step reachable with $N \geq n$ if and only if

$$\text{rank } \Gamma_N(a,b) = N + n - 1 \quad \text{at all times}$$

where a(z) and b(z) are given by (6.4).

PROOF. This proof is similar to that of Theorem 6.5. First, we partition the resultant matrix $\Gamma_N(a,b)$ as follows:

$$\Gamma_N(a,b) = \begin{array}{cc} \begin{array}{cc} N-1 & N \end{array} & \\ \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_3 & \Gamma_4 \end{bmatrix} & \begin{array}{c} n \\ N-1 \end{array} \end{array}$$

Postmultiply $\Delta = \begin{bmatrix} \Delta_1 & I_{N-1} \\ I_N & 0 \end{bmatrix}$ on both sides of the above equation,

where $\Delta_1 = -\Gamma_3^{-1}\Gamma_4$. Then we have

$$\Gamma_N(a,b)\Delta = \begin{bmatrix} \Gamma_1\Delta_1 + \Gamma_2 & \Gamma_1 \\ 0 & \Gamma_3 \end{bmatrix}$$

It can be shown that $\Gamma_1\Delta_1 + \Gamma_2 = C_N$ by comparing columns on both sides, where

$$C_N = [g \quad S_F g \; . \; . \; . \; S_F^{N-1} g] \quad .$$

Since rank $\Gamma_3 = N-1$ and rank $\Delta = 2N-1$ at all times, rank $\Gamma_1\Delta_1 + \Gamma_2 = n$ at all times if and only if rank $\Gamma_N = N + n - 1$ at all times. Therefore, the theorem is proved. []

Now let us consider the MIMO case. First, we define a time-varying version of ROWE's [1972] resultant matrix. Later we will find a generalized resultant matrix which is analogous to that of ANDERSON and JURY [1976].

Let polynomial matrices with time-varying coefficients, $Q(z)$ and $R(z)$, be given by

$$
\begin{aligned}
Q(z) &= z^n I + z^{n-1} Q_{n-1}(k) + \ldots + z Q_1(k) + Q_0(k) \\
R(z) &= z^{n-1} R_{n-1}(k) + \ldots + z R_1(k) + R_0(k)
\end{aligned}
\tag{6.9}
$$

where Q_i are $p \times p$ and R_i are $p \times m$ time-varying matrices.

(6.10) <u>DEFINITION</u>. The time-varying resultant matrix $\Gamma_n(Q,R)$ of two polynomial matrices $Q(z)$ and $R(z)$ given by (6.9) is defined as

$$
\Gamma_n(Q,R) = \begin{bmatrix}
Q_0 & 0 & \cdots & & R_0 & 0 & \cdots & \\
Q_1 & \sigma Q_0 & & & R_1 & \sigma R_0 & & \\
\vdots & & \ddots & & \vdots & & \ddots & \\
 & & & \sigma^{np-2} Q_0 & & & & \\
Q_{n-1} & & & & R_{n-1} & & & \sigma^{np-1} R_0 \\
I & \sigma Q_{n-1} & & \vdots & 0 & \sigma R_{n-1} & & \vdots \\
\vdots & I & \ddots & & \vdots & & \ddots & \\
 & & & \sigma^{np-2} Q_{n-1} & & & & \\
0 & \cdots & & I & 0 & \cdots & & \sigma^{np-1} R_{n-1}
\end{bmatrix}
$$

where $\Gamma_n(Q,R) \in A^{(np + n - 1)p \times \{(np-1)p + npm\}}$. []

Suppose that the MIMO linear time-varying discrete-time system is given by the input/output difference equation

$$y(k+p_1) + \sum_{i=0}^{p_1-1} Q_i(k)y(k+i) = \sum_{i=0}^{p_2-1} R_i(k)u(k+i) \tag{6.11}$$

where $p_1 > p_2$, $Q_i \in A^{p \times p}$, and $R_i \in A^{p \times m}$. Then the n-dimensional observable realization (F,G,H) of the above equation is given by (4.12a,b), where $n = p_1 p$. Define polynomial matrices related to (6.11) as follows :

$$\begin{aligned} Q(z) &= z^{p_1} I + z^{p_1-1} Q_{p_1-1}(k-p_1+1) + \ldots + zQ_1(k-1) + Q_0(k) \\ R(z) &= z^{p_2-1} R_{p_2-1}(k-p_2+1) + \ldots + zR_1(k-1) + R_0(k) \quad . \end{aligned} \tag{6.12}$$

Then we have the following theorem.

(6.13) <u>THEOREM</u>. The n-dimensional observable realization (F,G,H) given by (4.12a,b) with $n = p_1 p$ is n-step reachable at all times if and only if

$$\text{rank } \Gamma_{p_1}(Q,R) = (p_1 p + p_1 - 1)p \quad \text{at all times} ,$$

where Q(z) and R(z) are given by (6.12). []

We omit the proof of the above theorem since it is almost identical to that of Theorem 6.5. As in the SISO case, we can define the N-step time-varying resultant matrix in the MIMO case.

(6.14) DEFINITION. For an integer $N > p$ the time-varying resultant matrix $\Gamma_N(Q,R)$ of two polynomial matrices $Q(z)$ and $R(z)$ given by (6.9) is defined as

$$\Gamma_N(Q,R) = \begin{bmatrix} Q_0 & 0 & \cdots & & R_0 & 0 & \cdots & \\ Q_1 & \sigma Q_0 & & & R_1 & \sigma R_0 & & \\ \vdots & & \ddots & & & & \ddots & \\ & & & \sigma^{N-2}Q_0 & \vdots & & & \\ Q_{n-1} & & & & R_{n-1} & & & \sigma^{N-1}R_0 \\ I & \sigma Q_{n-1} & & \vdots & 0 & \sigma R_{n-1} & & \\ & I & \ddots & & & & & \vdots \\ \vdots & & \ddots & \sigma^{N-2}Q_{n-1} & \vdots & & & \\ 0 & & \cdots & I & 0 & \cdots & & \sigma^{N-1}R_{n-1} \end{bmatrix}$$

where $\Gamma_N(Q,R) \in A^{(Np + n-1)p \times \{(Np-1)p + Npm\}}$. []

(6.15) THEOREM. The n-dimensional observable realization (F,G,H) given by (4.12a,b) with $N > n = p_1 p$ is N-step reachable at all times if and only if

$$\operatorname{rank} \Gamma_N(Q,R) = (Np + n - 1)p \quad \text{at all times} ,$$

where $Q(z)$ and $R(z)$ are given by (6.12). []

The shortcoming of the time-varying ROWE's resultant is that the size of the matrix is too big to handle. Therefore, we need to reduce the size of the resultant matrix. In the time-invariant case, ANDERSON and JURY [1976] have defined a generalized Sylvester's resultant matrix. We will obtain the similar generalized resultant matrix in the time-varying case.

(6.16) <u>DEFINITION</u>. For each integer $i > 0$, a generalized time-varying resultant matrix of two polynomial matrices $Q(z)$ and $R(z)$ given by (6.9) is defined as

$$\Gamma^i(Q,R) = \begin{bmatrix} I & 0 & \cdots & 0 & \cdots & \sigma^{i-1}R_{n-1} \\ \sigma^{i-2}Q_{n-1} & I & & 0 & & \\ \vdots & & \ddots & \vdots & & \vdots \\ & & & I \;\; 0 & \sigma R_{n-1} & \\ & & & R_{n-1} & & \\ & & & & \vdots & \sigma^{i-1}R_0 \\ \sigma^{i-2}Q_0 & & & \vdots \;\; \vdots & \cdots & 0 \\ \vdots & \ddots & & & & \vdots \\ 0 & \cdots & & Q_0 \;\; R_0 & \cdots & 0 \end{bmatrix}$$

where $\Gamma^i(Q,R) \in A^{(i + n - 1)p \times \{(i - 1)p + im\}}$.

Using the above definition, we get a necessary and sufficient condition for n-step reachability of the given system.

(6.17) <u>THEOREM</u>. The n-dimensional observable realization (F,G,H) given by (4.12a,b) with $n = p_1 p$ is n-step reachable at all times if and only if there exists an integer $\alpha \leq np$ such that

(i) $\Gamma^\alpha(Q,R)$ has full rank at all times

(ii) rank $\Gamma^{\alpha+1}(Q,R)$ - rank $\Gamma^\alpha(Q,R) = p$ at all times.

The proof of the above theorem can be done by a slight modification of that of ANDERSON and JURY [1976]. Reachability of the system (F,G,H) can be checked by examining the ranks of generalized resultant matrices of growing size. Therefore, the computation may be reduced when the generalized resultant matrix is used for checking reachability of the system. we should note that if $\alpha > n$, then Definition 6.16 is exactly same as ROWE's resultant.

CHAPTER SEVEN
APPLICATIONS OF TIME-VARYING RESULTANT MATRICES

Having defined the discrete-time time-varying version of the resultant matrix in the previous chapter, we now continue the development of the resultant theory in the time-varying case. In the following, we shall consider coprimeness of polynomials and polynomial matrices with time-varying coefficients, a time-varying version of Bezout polynomial factorization, and compensator design of time-varying systems using the resultant matrix.

Let a pair of polynomials with time-varying coefficients, $a(z)$ and $b(z)$, be given by

$$
\begin{aligned}
a(z) &= z^n + z^{n-1} a_{n-1}(k) + \ldots + za_1(k) + a_0(k) \\
b(z) &= z^{n-1} b_{n-1}(k) + \ldots + zb_1(k) + b_0(k)
\end{aligned}
\tag{7.1}
$$

If there is a polynomial $c(z) \in A[z]$ such that $\deg c(z) < n$ and for some polynomials $\bar{a}(z) \in A[z]$ and $\bar{b}(z) \in A[z]$, $a(z) = c(z)\bar{a}(z)$ and $b(z) = c(z)\bar{b}(z)$, then the polynomial $c(z)$ is said to be <u>a left common factor</u> of two polynomials $a(z)$ and $b(z)$. If $a(z)$ and $b(z)$ do not have a left common factor, $a(z)$ and $b(z)$ are said to be left coprime (or relatively left prime). Then we have the following result.

(7.2) THEOREM. Two polynomials with time-varying coefficients, $a(z)$ and $b(z)$, given by (7.1) have a left common factor if and only if

$$\det \Gamma_n(a,b) = 0 \text{ at all times}$$

where $\Gamma_n(a,b)$ is the resultant matrix of $a(z)$ and $b(z)$ defined by (6.2).

PROOF. Suppose that the polynomials $a(z)$ and $b(z)$ given by (7.1) have a left common factor $c(z)$ of degree $0 < r < n-1$. Then $c(z)$ must be monic since $a(z)$ is monic. Let

$$a(z) = c(z)\bar{a}(z) \quad , \quad b(z) = c(z)\bar{b}(z)$$

where $\bar{a}(z)$, $\bar{b}(z)$, $c(z)$ are given by

$$\bar{a}(z) = z^{n-r} + z^{n-r-1}\bar{a}_{n-r-1}(k) + \ldots + \bar{a}_0(k)$$

$$\bar{b}(z) = z^{n-r-1}\bar{b}_{n-r-1}(k) + \ldots + \bar{b}_0(k)$$

$$\bar{c}(z) = z^{r} + z^{r-1}c_{r-1}(k) + \ldots + c_0(k) \ .$$

Then we have

$$\Gamma_n(a,b) = \begin{bmatrix} c_0 & 0 & \cdots & & & \\ c_1 & \sigma c_0 & & & & \\ & & \ddots & & & \\ \vdots & & \ddots & \cdots & & \\ c_{r-1} & & & & & \\ 1 & \sigma c_{r-1} & & & & \\ & \ddots & & & & \\ & & & 1 & & \end{bmatrix} \begin{bmatrix} \bar{a}_0 & 0 & & \bar{b}_0 & 0 & \cdots \\ \bar{a}_1 & \sigma\bar{a}_0 & & \bar{b}_1 & \sigma\bar{b}_0 & \\ \vdots & & \ddots & & & \ddots \\ \bar{a}_{n-r-1} & & \vdots & \bar{b}_{n-r-1} & & \\ 1 & \ddots & & \vdots & & \ddots \\ & & 1 & & & \end{bmatrix}$$

By Sylvester's inequality, rank $\Gamma_n(a,b) \leq (2n - r - 1)$ at all times. Therefore, det $\Gamma_n(a,b) = 0$ at all times.

Conversely, suppose that det $\Gamma_n(a,b) = 0$ at all times. Then there always exists (2n-1)-element column vector $x \neq 0$ such that

$$\Gamma_n(a,b)x = 0 \quad \text{at all times.}$$

Let $x = [c_0 \quad c_1 \ \cdots \ c_{n-2} \quad d_0 \quad d_1 \ \cdots \ d_{n-1}]'$ and define

$$c(z) = z^{n-2} c_{n-2} + \cdots + zc_1 + c_0$$
$$d(z) = z^{n-1} d_{n-1} + \cdots + zd_1 + d_0 \quad .$$

Then we have

$$a(z)c(z) + b(z)d(z) = 0$$

Since $a(z)$ is monic, it is invertible over $A((z^{-1}))$. Hence we get

$$c(z) = -a^{-1}(z)b(z)d(z) \quad .$$

The polynomial $b(z)d(z)$ must have $a(z)$ as a left factor since $c(z)$ is a polynomial. However, degrees of $b(z)$ and $d(z)$ are $n-1$ and $n-1$, respectively. Therefore, $a(z)$ and $b(z)$ must have a left common factor of degree at least 1. []

Combining the above result and Theorem 6.5, we see that the observable realization (F,g,h) of given polynomials $a(z)$ and $b(z)$ is not reachable if $a(z)$ and $b(z)$ have a left common factor. But, even if (F,g,h) is not reachable, $a(z)$ and $b(z)$ may not have a common factor.

Ilchmann et al. [1984] have defined a notion of coprimeness of two time-varying polynomials and polynomial matrices in terms of the time-varying version of Bezout polynomial factorization. However, this definition of coprimeness is not related to a notion of the resultant. In the following, we derive the time-varying version of Bezout identity from the reslutant matrix, and we relate this result to coprimeness of time-varying polynomials.

(7.3) <u>THEOREM</u>. Let the polynomials $a(z)$ and $b(z)$ be given by (7.1). Then there exist polynomials $c(z)$ and $d(z)$ such that

$$a(z)c(z) + b(z)d(z) = 1 \tag{7.4}$$

where deg $c(z) \leq n-2$, deg $d(z) \leq n-1$ if and only if

$$\det \Gamma_n(a,b) \neq 0 \quad \text{at all times.}$$

PROOF. Let $c(z)$ and $d(z)$ be denoted by

$$c(z) = z^{n-2}c_{n-2} + \ldots + zc_1 + c_0$$
$$d(z) = z^{n-1}d_{n-1} + \ldots + zd_1 + d_0$$

If we rewrite (7.4) in a matrix format, then we obtain

$$[c_0 \ldots c_{n-2} \quad d_0 \ldots d_{n-1}]\, \Gamma_n'(a,b) = [1 \quad 0 \ldots 0] \quad .$$

The above equation has a solution if and only if $\det \Gamma_n(a,b) \neq 0$ at all times, and thus the theorem follows. []

(7.5) COROLLARY. Let the polynomials $a(z)$, $b(z)$ be given by (7.1). Then for some integer $N > n$ there exist the polynomials $c(z)$ and $d(z)$ such that

$$a(z)c(z) + b(z)d(z) = 1 \tag{7.6}$$

where deg $c(z) \leq N-2$, deg $d(z) \leq N-1$ if and only if rank $\Gamma_N(a,b) = N + n - 1$ at all times. []

From the above results, we see that the existence of a Bezout identity is necessary but not sufficient for coprimeness of two polynomials.

(7.7) EXAMPLE. Consider the polynomials a(z) and b(z) given by

$$a(z) = z^2 + zk + k$$
$$b(z) = z$$

Then we have the resultant matrix $\Gamma_2(a,b)$ given by

$$\Gamma_2(a,b) = \begin{bmatrix} k & 0 & 0 \\ k & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Since det $\Gamma_2(a,b) \neq 0$ for all integers k except k = 0, a(z) and b(z) are left coprime. However, for any integer $N > 2$, rank $\Gamma_N(a,b) \neq N + 2 - 1$ at all times. Therefore, a(z) and b(z) does not satisfy a Bezout identity. []

Having studied the results of coprimeness and Bezout identity in the SISO case, we now consider the MIMO case.

Let polynomial matrices with time-varying coefficients, Q(z) and

$R(z)$, be given by

$$Q(z) = z^n I + z^{n-1}Q_{n-1} + \ldots + zQ_1 + Q_0$$

(7.8)

$$R(z) = z^{n-1}R_{n-1} + \ldots + zR_1 + R_0$$

where $Q_i \in A^{p\times p}$ and $R_i \in A^{p\times m}$. Here, we assume that $p \leq m$.

(7.9) THEOREM. The polynomial matrices $Q(z)$ and $R(z)$ given by (7.8) have a left common factor if and only if for any integer $i > 0$

$$\text{rank } \Gamma^i(Q,R) < (i+n-1)p \quad \text{at all times} \qquad []$$

(7.10) THEOREM. Let the polynomial matrices $Q(z)$ and $R(z)$ be given by (7.8). Then for some integer $i \geq 2$ there exist polynomial matrices $C(z)$ and $D(z)$ such that

$$Q(z)C(z) + R(z)D(z) = I$$

where $\deg C(z) \leq i-2$, $\deg D(z) \leq i-1$ if and only if

$$\text{rank } \Gamma^i(Q,R) = (i + n - 1)p \quad \text{at all times} \qquad []$$

As we have mentioned in the SISO case, the existence of Bezout identity implies the coprimeness, but the converse is not always true. This situation may occur since even if the polynomial matrices $Q(z)$ and

$R(z)$ are left coprime, the generalized resultant matrix $\Gamma^i(Q,R)$ for any integer i may not have full rank at all times.

From the above results and Theorem 6.17, we see that the n-dimensional observable realization (F,G,H) of two polynomial matrices $Q(z),R(z)$ given by (6.12) is N-step reachable with $N > n$ if and only if there exist the polynomial matrices $C(z)$ and $D(z)$ such that

$$Q(z)C(z) + R(z)D(z) = I$$

where $\deg C(z) < N-2$, $\deg D(z) < N-1$. This result corresponds to that of POOLLA [1984].

We now apply the resultant matrix to the design of output feedback controllers and obtain a result on a type of "assignability".

Again suppose that the system is given by the input/output difference equation

$$y(k+p_1) + \sum_{i=0}^{p_1-1} Q_i(k)y(k+i) = \sum_{i=0}^{p_2-1} R_i(k)u(k+i) \tag{7.11}$$

where $p > p_2$, $Q_i \in A^{p\times p}$, $R_i \in A^{p\times m}$. The polynomial matrices related to this system is defined by

$$\begin{aligned} Q(z) &= z^{p_1}I + z^{p_1-1}Q_{p_1-1}(k-p_1+1) + \ldots + zQ_1(k-1) + Q_0(k) \\ R(z) &= z^{p_2-1}R_{p_2-1}(k-p_2+1) + \ldots + zR_1(k-1) + R_0(k) \quad . \end{aligned} \tag{7.12}$$

Since $Q(z)$ is invertible over $A((z^{-1}))$, we can define $W(z)$ as

$$W(z) = Q^{-1}(z)R(z) \quad . \tag{7.13}$$

Then $W(z)$ is said to be the transfer-function matrix of the input/output difference equation (7.11), and (7.13) is called the left polynomial matrix-fraction representation of (7.11) (see KAMEN et al. [1985], and POOLLA [1984]). If for some matrices $\bar{Q}(z)$ and $\bar{R}(z)$, a transfer-function matrix $W(z)$ is given by

$$W(z) = \bar{R}(z)\bar{Q}^{-1}(z) \quad ,$$

then it is called the right polynomial matrix-fraction representation. If we let $U(z)$,$Y(z)$ be the polynomial representation of input and output sequences, then we have

$$Y(z) = W(z)U(z) \quad . \tag{7.14}$$

Consider the feedback control system shown in Fig 7.1. Here, $W_c(z)$ is the transfer function of the compensator to be designed, which has the form

$$W_c(z) = C(z)D^{-1}(z) \quad .$$

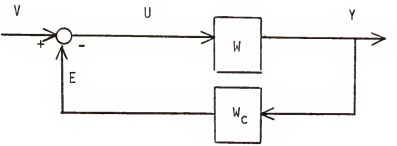


Fig 7.1 Feedback control system structure.

In the following, we design a compensator $W_c(z)$ such that the closed-loop system has preassigned dynamics.

Define

(7.15) $$E(z) = W_c(z)Y(z)$$

and let $\xi(z)$ be a vector such that

(7.16 a) $$D(z)\xi(z) = Y(z)$$

(7.16 b) $$E(z) = C(z)\xi(z)$$

The polynomial vector $\xi(z)$ is called the partial state of the system. From (7.13) and (7.14), we have

(7.17) $$Q(z)Y(z) = R(z)U(z).$$

Let $V(z)$ be a reference input. Then we obtain

$$V(z) = U(z) + E(z)$$

Premultiplying $R(z)$ on both sides of the above equation and from (7.16), (7.17), we obtain

$$\begin{aligned} R(z)V(z) &= R(z)U(z) + R(z)E(z) \\ &= Q(z)Y(z) + R(z)C(z)\xi(z) \\ &= (Q(z)D(z) + R(z)C(z))\xi(z). \end{aligned}$$

Inserting the above equation into (7.16a), we can get

$$Y(z) = D(z)(Q(z)D(z) + R(z)C(z))^{-1}R(z)V(z) \quad . \tag{7.18}$$

Let us denote $\Pi(z) = Q(z)D(Z) + R(z)C(z)$. Then we see that the closed-loop system dynamics are determined by the polynomial matrix $\Pi(z)$.

We now consider assigning the coefficients of the polynomial matrix $\Pi(z)$ by choosing $C(z)$ and $D(z)$.

Suppose that the n-dimensional observable realization (F,G,H) of the polynomials given by (7.12) with $n = p_1 p$ is N-step reachable at all times with $N > n$. Then from Theorem 6.17, there exists an integer $\alpha > 0$ such that

(i) $\Gamma^{\alpha}(Q,R)$ has full rank at all times

(ii) rank $\Gamma^{\alpha+1}(Q,R)$ - rank $\Gamma^{\alpha}(Q,R) = p$ at all times.

Let the polynomial matrix $\Pi(z)$ be given by

$$\Pi(z) = z^{\nu}I + z^{\nu-1}\Pi_{\nu-1} + \ldots + z\Pi_1 + \Pi_0$$

where $\nu = (\alpha + n - 1)$.

Since $\Gamma^{\alpha}(Q,R)$ has full rank at all times, there always exist matrices $D_0, D_1, \ldots, D_{\alpha-2} \in A^{m\times m}$, $C_0, C_1, \ldots, C_{\alpha-1} \in A^{p\times m}$ such that

$$\Gamma^{\alpha}(Q,R)\begin{bmatrix} D_{\alpha-2} \\ \vdots \\ D_0 \\ C_0 \\ \vdots \\ C_{\alpha-1} \end{bmatrix} = \begin{bmatrix} \Pi_{\nu-1} \\ \vdots \\ \\ \\ \\ \Pi_0 \end{bmatrix} \tag{7.19}$$

Let us define

$$C(z) = z^{\alpha-1}C_{\alpha-1} + \ldots + zC_1 + C_0$$
$$D(z) = z^{\alpha-1}I + z^{\alpha-2}D_{\alpha-2} + \ldots + zD_1 + D_0$$

Then from (7.19) we have

$$Q(z)C(z) + R(z)D(z) = \Pi(z)$$

Therefore, if the system (F,G,H) is N-step reachable, we can find the compensator $W_c(z) = C(z)D^{-1}(z)$ which results in a closed-loop system with preassigned dynamics. The realization of this compensator, which is called the controllable realization (see FUHRMANN [1976]), is derived in the appendix.

CHAPTER EIGHT
CONCLUDING REMARKS

By introducing the augmentation approach into the theory of discrete-time time-varying systems we have been able to design feedback control systems for a broad class of time-varying systems in the first part of this dissertation. Via augmentation, we can turn an N-step reachable time-varying system into a system with a fixed basis, which is a very restricted class of time-varying systems. Using this fixed basis, we were able to derive a generalized control canonical form and the cyclizability result. These results then led to the design of feedback control systems which result in a closed-loop system matrix that is algebraically equivalent to a constant matrix with arbitrary assignable eigenvalues. This assignability result corresponds to past work (see KAMEN et al. [1985] and POOLLA [1984]), but our results yield controllers having a much lower order in general than the controller derived by KAMEN et al. [1985].

In the second part of this dissertation , we developed the resultant theory of time-varying systems. From the discrete-time time-varying version of the resultant matrix we were able to obtain a test for coprimeness of polynomials and polynomial matrices with time-varying coefficients, to derive a Bezout identity, and to develop a compensator design based on a polynomial model. Moreover, we could relate

reachability of a system to the rank of the resultant, which resembles results in the time-invariant case.

There are many open problems which deserve investigation in the area of time-varying systems. We expect that the theories developed in this dissertation will be useful in solving some of these problems. We outline some of these below .

a) Continuous time case: We have not treated the continuous-time case in this dissertation. However, the augmentation approach may be applied to the continuous-time case easily. For example, WOLOVICH [1968] treated systems having a fixed lexicographic basis, which belong to a subset of the class of uniformly controllable time-varying systems. But, if we use the augmentation approach, then it is possible to design feedback control systems for uniformly controllable systems since the augmentation results in a system with a fixed basis. Actually, we can design feedback systems for a broad class of time-varying systems including uniformly controllable systems. We should note that the semilinear transformation which plays an important role in our development has a different form in the continuous-time case. The reader may refer to WOLOVICH [1968] and BUCY [1968].

b) Robustness: In the time-invariant case, even if the closed-loop system is stable, the compensator itself may not be stable (see SHAW [1971]). This may affect the sensitivity of the system. A similar situation may occur when the compensator for time-varying systems is designed by the augmentation approach. However, our constructive method for the augmentation is very flexible and gives infinitely many

different matrices for the augmentation. Therefore, by choosing an appropriate augmentation, instability of the compensator may be eliminated.

c) Tracking and disturbance rejection: We have not treated the problem of tracking and disturbance rejection in this dissertation. However, we believe that the augmentation approach would be very useful in studying this problem. It is very interesting that in the time-invariant case the most general robust controller which regulates the system with disturbances must have an augmented structure (see DAVISON and GOLDENBERG [1976]).

d) Theory of the resultant: In the time-invariant case, the resultant matrix is deeply related to the stability problem (see JURY [1976]). Therefore, as an extended project, one might consider the relationship between the resultant matrix and stability in the time-varying case. It would be very useful if we can extract the pole-zero structure of time-varying systems from the resultant matrix. The reader may refer to KAMEN [1986] for the pole-zero structure of time-varying systems. Other topics which one might consider are the development of an algorithm for finding common factors of polynomials and polynomial matrices with time-varying matrices and to extend the resultant theory of the discrete-time case to the continuous-time case.

e) Systems over rings: We can apply the augmentation approach to the construction of dynamic state-feedback for linear systems over commutative rings, if such commutative rings have the same property as Proposition 3.4. For example, EMRE and KHARGONEKAR [1982] have shown

that coefficient-assignment for a finite-free linear system over a commutative ring can be achieved by dynamic feedback compensators. If the augmentation approach is used to the construction of compensators for such systems, then we can get the same result. Moreover, our compensator has a much lower order than the compensator derived by EMRE and KHARGONEKAR [1982]. For more details, see KAMEN and HWANG [1986].

APPENDIX
CONTROLLABLE REALIZATION OF TIME-VARYING SYSTEMS

In Chapter Seven, we obtained a compensator that is described by a right polynomial matrix-fraction representation. From this form, we will obtain the controllable realization of the system in this appendix. This is the discrete-time time-varying version of FUHRMAN [1976]'s standard controllable realization. For the time-invariant case, the reader may refer to FUHRMAN [1976] and BROCKETT [1970].

Let the time-varying transfer function be given by

$$W(z) = C(z)D^{-1}(z) \tag{A.1}$$

where $C(z) \in A[z]^{p \times m}$ and $D(z) \in A[z]^{m \times m}$. In the following, we denote $C(z)$ and $D(z)$ as

$$\begin{aligned} C(z) &= z^{n-1}C_{n-1} + \ldots + zC_1 + C_0 \\ D(z) &= z^{n}I + z^{n-1}D_{n-1} + \ldots + zD_1 + D_0 \end{aligned} \tag{A.2}$$

By a power series expansion (see Kamen et al. [1985]), we can get

$$W(z) = z^{-1}W_1 + z^{-2}W_2 + \ldots + z^{-n}W_n + \ldots \tag{A.3}$$

where $W_i \in A$.

(A.4) PROPOSITION. (KAMEN et al. [1985]) Let (F,G,H) be a given time-varying system. Then the transfer-function matrix W(z) of the system is given by

$$W(z) = H(zI - F)^{-1}G \qquad []$$

Define the controllable realization (F,G,H) of the transfer function W(z) given by (A.1) and (A.2) as

$$\text{(A.5)} \qquad F = \begin{bmatrix} 0 & \cdots & 0 & -D_0 \\ I & \cdots & 0 & -D_1 \\ & \ddots & & \vdots \\ & & I & -D_{n-1} \end{bmatrix}, \quad G = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$H = [\sigma W_1 \;\; \sigma^2 W_2 \;\; \cdots \;\; \sigma^n W_n]$$

Then from Proposition A.4 we must show that the realization (F,G,H) given by (A.5) satisfies

$$W(z) = C(z)D^{-1}(z) = H(zI - F)^{-1}G \quad .$$

Define

$$\hat{X}(z) = (zI - F)^{-1}G$$

and let $\hat{X}'(z) = [\hat{X}'_1 \hat{X}'_2 \ldots \hat{X}'_n]$ where $\hat{X}_i \in A[z]^{m \times m}$. Then we have

$$\text{(A.6)} \qquad \begin{bmatrix} z\hat{X}_1 \\ z\hat{X}_2 \\ \vdots \\ z\hat{X}_n \end{bmatrix} - \begin{bmatrix} 0 & \cdots & 0 & -D_0 \\ I & & \vdots & -D_1 \\ & \ddots & & \vdots \\ & & I & -D_n \end{bmatrix} \begin{bmatrix} \hat{X}_1 \\ \hat{X}_2 \\ \vdots \\ \hat{X}_n \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

If we rewrite the above equation, then we obtain

$$\text{(A.7)} \qquad \begin{array}{ccccc} z\hat{X}_1 & + & D_0\hat{X}_n & = & I \\ z\hat{X}_2 & + & D_1\hat{X} & = & X_{n-1} \\ \vdots & & \vdots & & \vdots \\ z\hat{X}_n & + & D_{n-1}\hat{X}_n & = & \hat{X}_{n-1} \end{array}$$

By combining each equation in (A.7), we get

$$z^n\hat{X}_n + z^{n-1}D_{n-1}\hat{X}_n + \ldots + zD_1\hat{X}_n + D_0\hat{X}_n = I \quad .$$

This is equivalent to

$$\text{(A.8)} \qquad D(z)\hat{X}_n = I$$

From (A.7) and (A.8), we have

$$
\begin{aligned}
\hat{X}_n(z) &= D^{-1}(z) \\
\hat{X}_{n-1}(z) &= (zI - D_{n-1})D^{-1}(z) \\
&\vdots \\
\hat{X}_1(z) &= (z^{n-1}I + z^{n-2}D_{n-2} + \ldots + D_1)D^{-1}(z)
\end{aligned}
\tag{A.9}
$$

If we rewrite (A.9) in matrix form, then we obtain

$$
\hat{X}(z) = \begin{bmatrix} z^{n-1}I + z^{n-2}D_{n-1} + \ldots + D_1 \\ z^{n-2}I + z^{n-3}D_{n-1} + \ldots + D_2 \\ \vdots \\ zI + D_{n-1} \\ I \end{bmatrix} D^{-1}(z) \tag{A.10}
$$

We now consider $W(z) = C(z)D^{-1}(z)$. By comparing both sides of $W(z)D(z) = C(z)$, we have

$$
\begin{aligned}
C_{n-1} &= \sigma^n W_1 \\
C_{n-2} &= \sigma^n W_2 + (\sigma^{n-1}W_1)D_{n-1} \\
&\vdots \\
C_0 &= \sigma^n W_n + (\sigma^{n-1}W_{n-1})D_{n-1} + \ldots + (\sigma^2 W_2)D_2 + (\sigma W_1)D_1
\end{aligned}
\tag{A.11}
$$

By combining (A.10) and (A.11), we can get

$$
[\sigma W_1 \quad \sigma^2 W_2 \ \ldots \ \sigma^n W_n]\hat{X}(z) = C(z)D^{-1}(z)
$$

From the definition of $\hat{X}(z)$ and (A.5),

$$H(zI - F)^{-1}G = H\hat{X}(z)$$
$$= [\sigma W_1 \quad \sigma^2 W_2 \; . \; . \; . \; \sigma^n W_n]\hat{X}(z)$$

Therefore, $H(zI - F)^{-1}G = W(z)$, and thus (F,G,H) defined by (A.5) is the controllable realization of $W(z) = C(z)D^{-1}(z)$.

If the transfer function $W(z)$ is proper, i.e. if the power series expansion of $W(z)$ is given by

$$W(z) = W_0 + z^{-1}W_1 + \; . \; . \; .$$

then we have the controllable realization (F,G,H,J) such that F, G, H are defined by (A.5) and $J = W_0$.

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